

LOCAL SMOOTHING ESTIMATES NEAR A TRAPPED SET WITH INFINITELY MANY CONNECTED COMPONENTS

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ABSTRACT. We prove a local smoothing result for the Schrödinger equation on a class of surfaces of revolution which have infinitely many trapped geodesics. Our main result is a local smoothing estimate with loss (compared to [CM14]) depending on the accumulation rate of the critical points of the profile curve. The proof uses an h -dependent version of semiclassical propagation of singularities, and a result on gluing an h -dependent number of cutoff resolvent estimates.

1. INTRODUCTION

In this paper, we study the local smoothing effect for the Schrödinger equation on a class of manifolds with a trapped set which consists of infinitely many connected components. Local smoothing for the Schrödinger equation was first studied in [CS88], [Sjö87], and [Veg98]. By a result of [Doi96], the presence of trapping on an asymptotically Euclidean manifold necessitates a loss in the local smoothing estimate. Results in the presence of trapping have been obtained in [Bur04], [Chr07], [Chr08], [Chr11], [CW13], [Chr13], [CM14], and [Dat09], among others. Our main result is a generalization of the local smoothing estimate on \mathbb{R}^n

$$\int_0^T \|\langle x \rangle^{1/2+\epsilon} e^{it\Delta} u_0\|_{H^{1/2}}^2 dt \leq C \|u_0\|_{L^2}^2.$$

To describe our geometry, let $A \in C^\infty$ be such that

$$A^2(x) \sim \begin{cases} 1 + x^2, & x \sim 0, \\ \alpha_n^{-1} + (x - x_n)^m, & x \sim x_n \\ x^2, & |x| \rightarrow \infty, \end{cases}$$

where

$$x_n = 1 - 2^{-n}, \quad \alpha_n = 1 + 2^{-mn},$$

and $m \geq 3$ is odd, and A has no other critical points.

Our manifold is $X = \mathbb{R}_x \times \mathbb{R}_\theta / 2\pi\mathbb{Z}$ with the metric

$$ds^2 = dx^2 + A^2(x)d\theta^2.$$

Note that X is asymptotically Euclidean and has an infinite number of disjoint trapped sets. The trapping occurs $x = 0$ and at $x = x_n$ for each $n \geq 1$.

The early results in [Bur04], [Chr07], [Chr08], [Chr11], and [Dat09] involved non-degenerate hyperbolic trapping with a logarithmic loss in regularity. Later papers such as [CW13], [Chr13], and [CM14] consider degenerate trapping with polynomial loss in regularity. The novelty of this paper is allowing a trapped set

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with infinitely many connected components, all of which are degenerate. A new set of tools is developed, including an h -dependent version of semiclassical propagation of singularities and a result on gluing an h -dependent number of cutoff resolvent estimates.

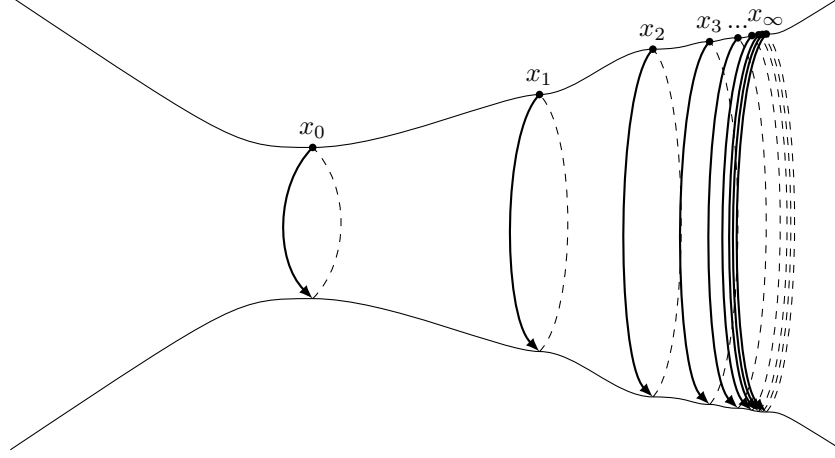


FIGURE 1. A piece of the manifold with trapped set at $x = x_0$, $x = x_1, \dots, x = x_\infty$.

Our main result is

Theorem 1. *Suppose X is as above, with $m \geq 1$ and assume u solves*

$$\begin{cases} (D_t - \Delta)u = 0 \text{ in } \mathbb{R} \times X, \\ u|_{t=0} = u_0 \in H^s, \end{cases}$$

for some $s > 0$ sufficiently large. Then for any $T < \infty, \epsilon > 0$, there exists a constant $C > 0$ such that

$$(1.1) \quad \int_0^T \|\langle x \rangle^{-1} \partial_x u\|_{L^2}^2 + \|\langle x \rangle^{-3/2} \partial_\theta u\|_{L^2}^2 dt \leq C(\|\langle D_\theta \rangle^{\beta(m)} u_0\|_{L^2}^2 + \|\langle D_x \rangle^{1/2} u_0\|_{L^2}^2),$$

where

$$\beta(m) = \frac{m^2}{(m-1)(m+2)} + \epsilon.$$

Remark 1.1. Note that we suffer a loss of $m/((m-1)(m+2))$ derivatives in the θ direction compared to [CM14] and no loss in the x (radial) direction. In fact this loss depends on how quickly the critical points x_n converge, as further examples will illustrate.

From the metric we obtain the volume form

$$dVol = A(x) dx d\theta$$

and the Laplace-Beltrami operator

$$\Delta f = (\partial_x^2 + A^{-2} \partial_\theta^2 + A^{-1} A' \partial_x) f.$$

We will conjugate Δ by an isometry and separate variables to reduce the analysis to a one dimensional semiclassical problem. Let $L : L^2(X, dV) \rightarrow (X, dxd\theta)$ be the isometry given by

$$Lu(x, \theta) = A^{1/2}u(x, \theta).$$

Then $\tilde{\Delta} = L\Delta L^{-1}$ is essentially self-adjoint on $L^2(X, dxd\theta)$, with mild assumptions on A . We find

$$-\tilde{\Delta} = -\partial_x^2 - A^{-2}\partial_\theta^2 + V_1(x),$$

where

$$V_1(x) = -\frac{1}{4}A^{-2}(A')^2 + \frac{1}{2}A^{-1}A''$$

We use a positive commutator argument to prove the smoothing estimate away from the critical points of A^2 . For the smoothing near critical points, we expand u into its Fourier series:

$$u(x, \theta) = \sum \varphi_k(x)e^{ik\theta}.$$

Then

$$\begin{aligned} (-\tilde{\Delta} - \lambda^2)u &= \sum (-\partial_x^2 - A^{-2}\partial_\theta^2 + V_1(x) - \lambda^2)\varphi_k(x)e^{ik\theta} \\ &= \sum e^{ik\theta}(-\partial_x^2 + A^{-2}k^2 + V_1(x) - \lambda^2)\varphi_k(x), \end{aligned}$$

so let

$$P_k(x) - \lambda^2 = -\partial_x^2 + A^{-2}k^2 + V_1(x) - \lambda^2.$$

We proceed by proving the local smoothing estimate for each k . By setting $h = k^{-1}$ and $z = h^2\lambda^2$, we obtain the semiclassical operator

$$P(h) - z = -h^2\partial_x^2 + V(x) - z$$

for

$$V(x) = A^{-2}(x) + h^2V_1(x).$$

Using a TT^* argument, we reduce the problem to proving a cutoff resolvent estimate for $P(h) - z$. In fact, since the subpotential V_1 is of lower (semiclassical) order, we need only prove a cutoff resolvent estimate for

$$-h^2\partial_x^2 + A^{-2}(x) - z.$$

This is proven by approximating the potential A^{-2} by a potential \tilde{V} which has a finite (but h -dependent) number of critical points. The key is to make sure the approximation is better than the spectral estimate. We prove our resolvent estimate on \tilde{V} by gluing together resolvent estimates near each critical point.

To glue together $N(h)$ resolvent estimates, we use a control theory version of propagation of singularities. Given two regions connected via the Hamiltonian flow, we control the L^2 mass in one by the L^2 in the other plus a term involving $P(h)$. By allowing the control region to grow in an h -dependent way, we obtain h -dependent constants in our estimate.

The propagation of singularities theorem allows us to control commutator terms supported away from the critical points of H_p in terms of operators supported far away from the trapping, where we have better estimates. The novel aspect is that we allow the number of critical points being glued to grow with h , which causes some loss in our final resolvent estimate.

Finally, this approach is applied to a variety of other example manifolds to demonstrate the relationship between the speed of convergence of the critical points and the loss in our resolvent estimate.

Remark 1.2. Note that throughout this paper C denotes a general purpose constant which may change from line to line.

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2. PRELIMINARIES

2.1. Geometry. Consider $S^1 \times \mathbb{R}$ with the metric

$$ds^2 = dx^2 + A^2(x)d\theta^2.$$

We require $A \geq \epsilon > 0$, and that the metric is a short range perturbation of \mathbb{R}^2 . The second condition means

$$|\partial^\alpha(A^2 - x^2)| \leq C_\alpha \langle x \rangle^{-2-|\alpha|},$$

as $|x| \rightarrow \infty$. A useful consequence of this is

$$(2.1) \quad |\partial_x^k A| \leq C_k \langle x \rangle^{1-k}.$$

Proof. For $k = 0$, the short range perturbation assumption tells us

$$x^2 - C_0 \langle x \rangle^{-2} \leq A^2 \leq x^2 + C_0 \langle x \rangle^{-2}$$

as $|x| \rightarrow \infty$. Thus, as $|x| \rightarrow \infty$ we have

$$c \langle x \rangle \leq A \leq C \langle x \rangle.$$

This proves the $k = 0$ case. To see how the higher k cases work, consider the $k = 1$ case. We have

$$A^{-1}(2x - C_1 \langle x \rangle^{-3}) \leq 2A' \leq A^{-1}(2x + C_1 \langle x \rangle^{-3}),$$

or

$$c' \leq \frac{2x - C_1 \langle x \rangle^{-3}}{C \langle x \rangle} \leq 2A' \leq \frac{2x + C_1 \langle x \rangle^{-3}}{c \langle x \rangle} \leq C'.$$

The higher cases follow similarly. Each time, we isolate the term of the form $A^{(k)}A$, divide both sides by A , and use the lower cases. \square

Remark 2.1. Our approach follows [CW13] and [CM14]. We apply a positive commutator argument to obtain local smoothing away from critical points. We then decompose u into high and low frequency parts and estimate the low frequency part in terms of the high frequency part. The high frequency part is then estimated using a TT^* argument and gluing.

We have

$$dVol = A(x) dx d\theta,$$

and

$$\Delta f = (\partial_x^2 + A^{-2} \partial_\theta^2 + A^{-1} A' \partial_x) f.$$

Define

$$T : L^2(X, dVol) \rightarrow L^2(X, dx d\theta)$$

by

$$Tu(x, \theta) = A^{1/2}(x)u(x, \theta)$$

and let $\tilde{\Delta} = T\Delta T^{-1}$. Then

$$\begin{aligned}
\tilde{\Delta}f &= T(\partial_x^2 + A^{-2}\partial_\theta^2 + A^{-1}A'\partial_x)(A^{-1/2}f) \\
&= T\left(\partial_x\left(-\frac{1}{2}A^{-3/2}A'f + A^{-1/2}\partial_x f\right) + A^{-5/2}\partial_\theta^2 f\right. \\
&\quad \left.+ A^{-1}A'\left(-\frac{1}{2}A^{-3/2}A'f + A^{-1/2}\partial_x f\right)\right) \\
&= T\left(\frac{3}{4}A^{-5/2}(A')^2f - \frac{1}{2}A^{-3/2}A''f - \frac{1}{2}A^{-3/2}A'\partial_x f\right. \\
&\quad \left.- \frac{1}{2}A^{-3/2}A'\partial_x f + A^{-1/2}\partial_x^2 f + A^{-5/2}\partial_\theta^2 f - \frac{1}{2}A^{-5/2}(A')^2f\right. \\
&\quad \left.+ A^{-3/2}A'\partial_x f\right) \\
&= (\partial_x^2 + A^{-2}\partial_\theta^2 - V_1(x))f,
\end{aligned}$$

where

$$-V_1(x) = \frac{1}{4}A^{-2}(A')^2 - \frac{1}{2}A^{-1}A''.$$

Note that $\tilde{\Delta}$ is essentially self-adjoint on $L^2(X, dx d\theta)$, as long as $A(x)$ satisfies our assumptions.

2.2. Positive Commutator Argument. We first prove local smoothing away from critical points of $A(x)$:

Lemma 2.2. *Let u be a solution to the Schrödinger equation on our manifold. Then for any $T < \infty$ there exists a constant C such that*

$$(2.2) \quad \int_0^T \|\langle x \rangle^{-1} \partial_x u\|_{L^2}^2 + \|\sqrt{x A'(x)} \langle x \rangle^{-2} \partial_\theta u\|_{L^2}^2 dt \leq C \|u_0\|_{H^{1/2}}.$$

Proof. Let $B = \arctan(x)\partial_x$. Then

$$\begin{aligned}
[\tilde{\Delta}, B] &= (\partial_x^2 + A^{-2}\partial_\theta^2 - V_1(x))B - B(\partial_x^2 + A^{-2}\partial_\theta^2 - V_1(x)) \\
&= -\frac{2x}{(x^2 + 1)^2}\partial_x + \frac{2}{1 + x^2}\partial_x^2 - \arctan(x)\partial_x(A^{-2})\partial_\theta^2 \\
&\quad + \arctan(x)\partial_x(V_1(x)) \\
&= -2x\langle x \rangle^{-4}\partial_x + 2\langle x \rangle^{-2}\partial_x^2 + 2\arctan(x)A^{-3}A'\partial_\theta^2 + \arctan(x)V_1'
\end{aligned}$$

Let $P = D_t - \tilde{\Delta}$. Then

$$\begin{aligned}
0 &= \int_0^T \langle BPu, u \rangle - \langle Bu, Pu \rangle dt \\
&= \int_0^T \langle [B, P]u, u \rangle dt - i \langle Bu, u \rangle \Big|_0^T \\
&= \int_0^T \langle [B, -\tilde{\Delta}]u, u \rangle dt + \langle \arctan(x)D_x u, u \rangle \Big|_0^T,
\end{aligned}$$

or

$$\int_0^T \langle [B, \tilde{\Delta}]u, u \rangle dt = \langle \arctan(x)D_x u, u \rangle \Big|_0^T.$$

First we will estimate the right hand side:

$$\begin{aligned}
|\langle \arctan(x) D_x u, u \rangle| &= |\langle \arctan(x) \langle D_x \rangle^{1/2} \langle D_x \rangle^{-1/2} D_x u, u \rangle| \\
&= |\langle \langle D_x \rangle^{-1/2} D_x u, \langle D_x \rangle^{1/2} (\arctan(x) u) \rangle| \\
&\leq C \|u\|_{H^{1/2}} \|\arctan(x) u\|_{H^{1/2}} \\
&\leq C \|u\|_{H^{1/2}}^2 \\
&= C \|u_0\|_{H^{1/2}}^2
\end{aligned}$$

where the second to last line follows from the fact that \arctan and its first derivative are bounded in L^∞ .

On the left hand side,

$$\begin{aligned}
\int_0^T \langle [B, \tilde{\Delta}] u, u \rangle dt &= \int_0^T \langle 2 \langle x \rangle^{-2} \partial_x^2 u, u \rangle + \langle 2 \arctan(x) A^{-3} A' \partial_\theta^2 u, u \rangle dt \\
&\quad - \int_0^T \langle x \langle x \rangle^{-4} \partial_x u, u \rangle - \langle \arctan(x) V_1' u, u \rangle dt.
\end{aligned}$$

We will move the second integral to the right, so we must bound it by $\|u\|_{H^{1/2}}^2$. Each term in $V_1'(x)$ consists of A raised to a negative power, multiplied some of its derivatives. Recall that $A \geq \epsilon$ to bound the negative powers of A . We bound the rest using the short range perturbation inequality (2.1). Thus $\arctan(x) V_1'(x)$ is bounded, so

$$\begin{aligned}
&\left| \int_0^T \langle x \langle x \rangle^{-4} \partial_x u, u \rangle - \langle \arctan(x) V_1' u, u \rangle dt \right| \\
&\leq C \int_0^T |\langle x \langle x \rangle^{-4} \langle D_x \rangle^{1/2} \langle D_x \rangle^{-1/2} \partial_x u, u \rangle| + \|u\|_{L^2}^2 dt \\
&= C \int_0^T \|u\|_{H^{1/2}} \|x \langle x \rangle^{-4} u\|_{H^{1/2}} + \|u\|_{L^2}^2 dt \\
&\leq C \sup_{0 \leq t \leq T} \|u\|_{H^{1/2}}^2 \\
&= C \|u_0\|_{H^{1/2}}^2.
\end{aligned}$$

Of the remaining part of the left hand side, we do the term involving x derivatives using integration by parts:

$$\begin{aligned}
\int_0^T \langle 2 \langle x \rangle^{-2} \partial_x^2 u, u \rangle dt &= -2 \int_0^T \langle \partial_x u, \partial_x (\langle x \rangle^{-2} u) \rangle dt \\
&= -2 \int_0^T \|\langle x \rangle^{-1} \partial_x u\|_{L^2}^2 + \langle \partial_x u, 2x \langle x \rangle^{-2} u \rangle dt.
\end{aligned}$$

Now if we move the second term to the right hand side we can bound it by $\|u_0\|_{H^{1/2}}^2$ as before.

The term involving ∂_θ is easy because everything commutes. If we combine everything we have so far, we get the estimate

$$\int_0^T \|\langle x \rangle^{-1} \partial_x u\|_{L^2}^2 + \|\sqrt{\arctan(x) A^{-3} A'} \partial_\theta u\|_{L^2}^2 dt \leq C \|u_0\|_{H^{1/2}}^2.$$

To make this more understandable, we first need a lower bound on $\arctan(x)A^{-3}$. Since $A \geq \epsilon$, the inequality (2.1) tells us $A^{-3} \geq C \langle x \rangle^{-3}$. For $x < 0$, $A' < 0$, and for $x > 0$, $A' > 0$, so $\arctan(x)A' = |\arctan(x)A'|$. Similarly, $xA' = |xA'|$. Then using the fact that $|x| \langle x \rangle^{-1} \leq C |\arctan(x)|$ we have

$$Cx A' \langle x \rangle^{-4} \leq \arctan(x) A^{-3} A'$$

and we obtain our estimate. \square

This tells us that we have perfect smoothing in the x direction, and we only lose smoothing in the θ direction near the critical points of A . To obtain the estimate near the critical points of A we decompose into high and low frequencies.

2.3. Frequency Decomposition. We decompose u into a Fourier series:

$$u(t, x, \theta) = \sum_k e^{ik\theta} u_k(t, x), \quad u_0(x, \theta) = \sum_k e^{ik\theta} u_{0,k}(x).$$

First note the 0 mode satisfies the Schrödinger equation:

$$(D_t - \tilde{\Delta})u_0(t, x) = (D_t - \tilde{\Delta}) \frac{1}{2\pi} \int u(t, x, \theta) d\theta = 0,$$

and $u_0(0, x) = u_{0,0}(x)$. Thus we can use the estimate (2.2) from the previous section:

$$\begin{aligned} \int_0^T (\| \langle x \rangle^{-1} \partial_x u_0 \|_{L^2} + \| \langle x \rangle^{-3/2} \partial_\theta u_0 \|_{L^2}^2 dt \\ = \int_0^T \| \langle x \rangle^{-1} \partial_x u_0 \|_{L^2}^2 dt \\ \leq \int_0^T \| \langle x \rangle^{-1} \partial_x u_0 \|_{L^2}^2 dt + \| \sqrt{x A'(x)} \langle x \rangle^{-2} \partial_\theta u_0 \|_{L^2}^2 dt \\ \leq C \| u_{0,0} \|_{H^{1/2}}^2 \end{aligned}$$

Let $\chi \in C_0^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ near the critical points of A . Suppose that for each k

$$(2.3) \quad \int_0^T \| \chi(x) k u_k \|_{L_x^2}^2 dt \leq \| \langle k \rangle^{\alpha(A)} u_{0,k} \|_{L^2}^2 + \| u_{0,k} \|_{H^{1/2}}^2,$$

for some $\alpha(A) > 1/2$. Since $\partial_\theta(e^{ik\theta} u_k) = k e^{ik\theta} u_k$, this tells us

$$\begin{aligned} \int_0^T \| \langle x \rangle^{-3/2} \partial_\theta(e^{ik\theta} u_k) \|_{L_{x,\theta}^2}^2 dt \\ \leq C \int_0^T \| \langle x \rangle^{-3/2} \chi(x) k e^{ik\theta} u_k \|_{L_{x,\theta}^2}^2 + \| \langle x \rangle^{-3/2} (1 - \chi(x)) k e^{ik\theta} u_k \|_{L_{x,\theta}^2}^2 dt \\ \leq C \int_0^T \| \chi(x) k u_k \|_{L_x^2}^2 + \| \sqrt{x A'(x)} \langle x \rangle^{-2} k e^{ik\theta} u_k \|_{L_{x,\theta}^2}^2 dt \\ \leq C \left(\| \langle k \rangle^{\alpha(A)} u_{0,k} \|_{L^2}^2 + \| u_{0,k} \|_{H^{1/2}}^2 + \int_0^T \| \sqrt{x A'(x)} \langle x \rangle^{-2} k e^{ik\theta} u_k \|_{L_{x,\theta}^2}^2 dt \right) \end{aligned}$$

Then, by orthogonality,

$$\begin{aligned}
\int_0^T \|\langle x \rangle^{-3/2} \partial_\theta u\|_{L^2}^2 dt &= \sum_k \int_0^T \|\langle x \rangle^{-3/2} \partial_\theta (e^{ik\theta} u_k)\|_{L^2}^2 dt \\
&\leq C \sum_k \left(\|\langle k \rangle^{\alpha(A)} u_{0,k}\|_{L^2}^2 + \|u_{0,k}\|_{H^{1/2}}^2 \right. \\
&\quad \left. + \int_0^T \|\sqrt{x A'(x)} \langle x \rangle^{-2} k e^{ik\theta} u_k\|_{L^2_{x,\theta}}^2 dt \right) \\
&\leq C \sum_k \left(\|\langle D_\theta \rangle^{\alpha(A)} (e^{ik\theta} u_{0,k})\|_{L^2}^2 + \|\langle D_x \rangle^{1/2} (e^{ik\theta} u_{0,k})\|_{L^2}^2 \right) \\
&\quad + C \int_0^T \|\sqrt{x A'(x)} \langle x \rangle^{-2} \partial_\theta u\|_{L^2_{x,\theta}}^2 dt \\
&\leq C \left(\|\langle D_\theta \rangle^{\alpha(A)} u_0\|_{L^2}^2 + \|\langle D_x \rangle^{1/2} u_0\|_{L^2}^2 \right),
\end{aligned}$$

where the $\|u_0\|_{H^{1/2}}^2$ term coming from the integral is absorbed by the $\|\langle D_\theta \rangle^{\alpha(A)} u_0\|_{L^2}^2$ term. So it suffices to prove (2.3). Define

$$P_k = -D_x^2 - A^{-2}(x)k^2 - V_1(x)$$

and note that $(D_t + P_k)u_k = 0$. From here on, we will drop the k on u , and assume u is a solution to $(D_t + P_k)u = 0$

Let $\psi \in C_0^\infty(\mathbb{R})$ be an even function with $\psi = 1$ near 0, with small support. Define

$$u_{hi} = \psi(D_x/k)u, \quad u_{lo} = (1 - \psi(D_x/k))u.$$

Remark 2.3. The names u_{hi} and u_{lo} may seem backwards at first. To understand it, keep in mind that k is the frequency we are interested in. If k is small compared to D_x , then u_{hi} vanishes.

2.4. Low Frequency. We reduce the low frequency case to the high frequency case.

Lemma 2.4. *Suppose u is a solution to $(D_t + P_k)u = 0$. Then there exists $C > 0$ such that*

$$\int_0^T \|\langle x \rangle^{-1} k u_{lo}\|_{L^2}^2 dt \leq C(\|u_0\|_{H^{1/2}}^2 + \|k A^{-3/2} \tilde{\psi}(D_x/k)u\|_{L^2}^2),$$

where u_{lo} and u_{hi} are as above, and $\tilde{\psi} \in C_0^\infty$, with $\tilde{\psi} \equiv 1$ on the support of ψ .

Proof. Since $(D_t + P_k)u = 0$,

$$(D_t + P_k)u_{lo} = [D_t + P_k, \psi(D_x/k)]u = [P_k, \psi(D_x/k)]u.$$

Since D_x^2 and $\psi(D_x/k)$ are, on the frequency side, both multiplication, they commute, and the above is equal to

$$[-A^{-2}(x)k^2 - V_1(x), \psi(D_x/k)]u.$$

Note that $V_1(x)$ is an L^2 bounded pseudodifferential operator, so we write

$$[-V_1(x), \psi(D_x/k)] = L_1,$$

where we can bound $\|L_1\|$ independently of k .

For the remaining term, first note that $(A^{-2})^{(j)} \in S^0$, for $j \geq 1$. Using Taylor's theorem (omitting the remainder term, as we will absorb it into other terms), we have

$$\begin{aligned}
& [k^2 A^{-2}(x), \psi(D_x/k)]u \\
&= \frac{k^2}{2\pi} \iint_{\mathbb{R}^2} e^{i\langle x-y, \xi \rangle} \psi(\xi/k) (A^{-2}(x) - A^{-2}(y)) u(y) dy d\xi \\
&= \frac{k^2}{2\pi} \sum_{j=1}^N \iint_{\mathbb{R}^2} (x-y)^j e^{i\langle x-y, \xi \rangle} \psi(\xi/k) (A^{-2})^{(j)}(x) u(y) dy d\xi \\
&= \frac{k^2}{2\pi} \sum_{j=1}^N \iint_{\mathbb{R}^2} e^{i\langle x-y, \xi \rangle} k^{-j} D_\xi^{(j)} \psi(\xi/k) (A^{-2})^{(j)}(x) u(y) dy d\xi
\end{aligned}$$

The terms for $j \geq 2$ have non-positive powers of k , and so we can write

$$[k^2 A^{-2}(x), \psi(D_x/k)]u = (-ik(A^{-2})'(x)\psi'(D_x/k) + L_2)u,$$

where L_2 is a bounded pseudodifferential operator, with bound independent of k . Then all together we may write

$$(D_t + P_k)u_{10} = kA^{-3}(x)L\tilde{\psi}(D_x/k)u + L_3u,$$

where L, L_3 are bounded independently of k . Written in this way we can easily see the most important parts of the expression: The frequency cutoff $\tilde{\psi}$, the spatial decay given by A^{-3} , and the frequency k .

Next we use a positive commutator argument on u_{10} . Recall $B(x) = \arctan(x)\partial_x$ and note $B^* = -\langle x \rangle^{-2} - \arctan(x)\partial_x$. We have

$$\begin{aligned}
& 2i \operatorname{Im} \int_0^T \left\langle i^{-1} B(kA^{-3} L\tilde{\psi}(D_x/k) + L_3)u, u_{10} \right\rangle dt \\
&= 2i \operatorname{Im} \int_0^T \left\langle i^{-1} B(D_t + P_k)u_{10}, u_{10} \right\rangle dt \\
&= \int_0^T \left\langle i^{-1} B(D_t + P_k)u_{10}, u_{10} \right\rangle dt - \int_0^T \left\langle u_{10}, i^{-1} B(D_t + P_k)u_{10} \right\rangle dt \\
&= \int_0^T \left\langle i^{-1} B(D_t + P_k)u_{10}, u_{10} \right\rangle dt - \int_0^T \left\langle (i^{-1} B)^* u_{10}, (D_t + P_k)u_{10} \right\rangle dt \\
&= \int_0^T \left\langle i^{-1} B(D_t + P_k)u_{10}, u_{10} \right\rangle dt - \int_0^T \left\langle (i^{-1} B)u_{10}, (D_t + P_k)u_{10} \right\rangle dt \\
&\quad + \int_0^T \left\langle (i^{-1} B) - (i^{-1} B)^* u_{10}, (D_t + P_k)u_{10} \right\rangle dt \\
&= \int_0^T \left\langle [i^{-1} B, P_k]u_{10}, u_{10} \right\rangle dt - i \left\langle \arctan(x) D_x u_{10}, u_{10} \right\rangle \Big|_0^T \\
&\quad - \int_0^T \left\langle ((iB) - (iB)^*)u_{10}, (D_t + P_k)u_{10} \right\rangle dt \\
&= \int_0^T \left\langle [i^{-1} B, P_k]u_{10}, u_{10} \right\rangle dt - i \left\langle \arctan(x) D_x u_{10}, u_{10} \right\rangle \Big|_0^T \\
&\quad - \int_0^T \left\langle \langle x \rangle^{-2} u_{10}, (kA^{-3}(x)L\tilde{\psi}(D_x/k) + L_3)u \right\rangle dt.
\end{aligned}$$

or,

$$\begin{aligned}
\int_0^T \left\langle [i^{-1} B, P_k]u_{10}, u_{10} \right\rangle dt &= i \left\langle \arctan(x) D_x u_{10}, u_{10} \right\rangle \Big|_0^T + \\
&\quad + \int_0^T \left\langle \langle x \rangle^{-2} u_{10}, (kA^{-3}(x)L\tilde{\psi}(D_x/k) + L_3)u \right\rangle dt \\
&\quad + 2i \operatorname{Im} \int_0^T \left\langle i^{-1} B(kA^{-3} L\tilde{\psi}(D_x/k) + L_3)u, u_{10} \right\rangle dt.
\end{aligned}$$

We will bound the right hand side. First note that by averaging the derivative appearing in B over both u and u_{10} we obtain

$$|\langle BL_3 u, u_{10} \rangle| \leq C \|u_0\|_{H^{1/2}}^2.$$

so we may absorb the L_3 terms into the $H^{1/2}$ norm.

The remaining terms on the right hand side may be bounded similarly:

$$\begin{aligned}
& |\langle \langle x \rangle^{-2} u_{10}, (kA^{-3}(x)L\tilde{\psi}(D_x/k)u) | \\
&\leq \|A^{-3/2} \langle x \rangle^{-2} u_{10}\|_{L^2} \|kA^{-3/2} L\tilde{\psi}(D_x/k)u\|_{L^2} \\
&\leq \frac{1}{2} \left(\|A^{-3/2} \langle x \rangle^{-2} u_{10}\|_{L^2}^2 + \|kA^{-3/2} L\tilde{\psi}(D_x/k)u\|_{L^2}^2 \right) \\
&\leq \frac{1}{2} \left(C \|u\|_{L^2}^2 + \|kA^{-3/2} L\tilde{\psi}(D_x/k)u\|_{L^2}^2 \right).
\end{aligned}$$

and

$$\begin{aligned} \left| \left\langle i^{-1} B(kA^{-3} L \tilde{\psi}(D_x/k) u, u_{\text{lo}}) \right\rangle \right| &= \left| \left\langle kA^{-3/2} L \tilde{\psi}(D_x/k) u, A^{-3/2} B^* u_{\text{lo}} \right\rangle \right| \\ &\leq C \left(\|kA^{-3/2} L \tilde{\psi}(D_x/k) u\|_{L^2}^2 + \|A^{-3/2} B^* u_{\text{lo}}\|_{L^2}^2 \right). \end{aligned}$$

Then note

$$\begin{aligned} \|A^{-3/2} B^* u_{\text{lo}}\|_{L^2}^2 &\leq \|A^{-3/2} \langle x \rangle^{-2} u_{\text{lo}}\|_{L^2}^2 + \|A^{-3/2} \arctan(x) \partial_x (1 - \psi(D_x/k)) u\|_{L^2}^2 \\ &\leq C (\|u\|_{L^2}^2 + \|A^{-3/2} \partial_x u\|_{L^2}^2). \end{aligned}$$

Since $L \in \Psi^0$ and $A^{-3/2} \in \Psi^0$, we have $[A^{-3/2}, L] \in \Psi^{-1}$, so we may absorb the commutator term below into the other term:

$$\begin{aligned} \|kA^{-3/2} L \tilde{\psi}(D_x/k) u\|_{L^2}^2 &= \|kLA^{-3/2} \tilde{\psi}(D_x/k) u + k[A^{-3/2}, L] \tilde{\psi}(D_x/k) u\|_{L^2}^2 \\ &\leq C \|kA^{-3/2} \tilde{\psi}(D_x/k) u\|_{L^2}^2. \end{aligned}$$

Next we work on the left hand side. We have

$$\begin{aligned} [B, P_k] &= -2 \arctan(x) A^{-3}(x) A'(x) k^2 + 2 \langle x \rangle^{-2} \partial_x^2 \\ &\quad - \arctan(x) V_1(x) + 2x \langle x \rangle^{-2} \partial_x. \end{aligned}$$

The last two terms coming from the $[B, P_k]$ term may be absorbed into the $\|u_0\|_{H^{1/2}}^2$ term, and

$$|\langle \arctan(x) A^{-3}(x) A'(x) k^2 u_{\text{lo}}, u_{\text{lo}} \rangle| \leq \|kA^{-3/2} \tilde{\psi}(D_x/k) u\|_{L^2}^2.$$

Thus

$$\int_0^T \|\langle x \rangle^{-1} \partial_x u_{\text{lo}}\|_{L^2}^2 dt \leq C (\|u_0\|_{H^{1/2}}^2 + \|kA^{-3/2} \tilde{\psi}(D_x/k) u\|_{L^2}^2).$$

Then we have

$$\int_0^T \|\langle x \rangle^{-1} k u_{\text{lo}}\|_{L^2}^2 dt \leq C \int_0^T \|\langle x \rangle^{-1} \partial_x u_{\text{lo}}\|_{L^2}^2 dt,$$

so finally

$$\int_0^T \|\langle x \rangle^{-1} k u_{\text{lo}}\|_{L^2}^2 dt \leq C (\|u_0\|_{H^{1/2}}^2 + \|kA^{-3/2} \tilde{\psi}(D_x/k) u\|_{L^2}^2).$$

□

The next section will concern estimating the term $\|kA^{-3/2} \tilde{\psi}(D_x/k) u\|_{L^2}^2$.

2.5. High Frequency. We use a TT^* argument to reduce the high frequency estimate to a cutoff resolvent estimate.

Lemma 2.5. *Let $h = \frac{1}{|k|}$ and suppose*

$$(2.4) \quad \|\chi \tilde{\psi}((hD_x)^2 + V - z \pm i0)^{-1} \tilde{\psi} \chi\|_{L_x^2 \rightarrow L_x^2} \leq Ch^{-2(1-r)},$$

where $V = A^{-2}(x) + h^2 V_1(x)$. Then there exists $C > 0$ such that

$$\int_0^T \|\chi k \tilde{\psi}(D_x/k) u\|_{L^2}^2 dt \leq C \|k^r u_0\|_{L^2}^2,$$

where $\chi \equiv 1$ in a neighborhood of the critical points of A .

Proof. We consider the operator

$$F(t)g = \chi(x)\tilde{\psi}(D_x/k)k^r e^{-itP_k}g,$$

and determine for which r we have a bounded map $F : L_x^2 \rightarrow L^2([0, T]; L_x^2)$. We have that F is such a mapping if and only if

$$FF^* : L^2([0, T]; L_x^2) \rightarrow L^2([0, T]; L_x^2).$$

We have

$$\begin{aligned} \langle f(x), F^*g(t, x) \rangle_{L_x^2} &= \langle F(t)f(x), g(t, x) \rangle_{L^2([0, T]; L_x^2)} \\ &= \int_0^T \langle F(t)f(x), g(t, x) \rangle dt \\ &= \int_0^T \left\langle \chi(x)\tilde{\psi}(D_x/k)k^r e^{-itP_k}f(x), g(t, x) \right\rangle dt \\ &= \int_0^T \left\langle f(x), e^{itP_k}k^r \tilde{\psi}(D_x/k)\chi(x)g(t, x) \right\rangle dt \\ &= \left\langle f(x), k^r \int_0^T e^{itP_k}\tilde{\psi}(D_x/k)\chi(x)g(t, x) dt \right\rangle, \end{aligned}$$

so

$$F^*g(t, x) = k^r \int_0^T e^{itP_k}\tilde{\psi}(D_x/k)\chi(x)g(t, x) dt.$$

Thus

$$FF^*f(t, x) = \chi(x)\psi(D_x/k)k^{2r} \int_0^T e^{i(s-t)P_k}\tilde{\psi}(D_x/k)\chi(x)f(s, x) ds.$$

We split $FF^*f(t, x)$ up as

$$FF^*f(t, x) = \chi(x)\tilde{\psi}(D_x/k)(v_1 + v_2),$$

where

$$\begin{aligned} v_1 &= k^{2r} \int_0^t e^{i(s-t)P_k}\tilde{\psi}(D_x/k)\chi(x)f(s, x) ds \\ v_2 &= k^{2r} \int_t^T e^{i(s-t)P_k}\tilde{\psi}(D_x/k)\chi(x)f(s, x) ds. \end{aligned}$$

Next we calculate $(D_t + P_k)v_j$.

$$\begin{aligned} D_tv_1 &= D_t \left(k^{2r} e^{-itP_k} \int_0^t e^{isP_k}\tilde{\psi}(D_x/k)\chi(x)f(s, x) ds \right) \\ &= -P_k \left(k^{2r} e^{-itP_k} \int_0^t e^{isP_k}\tilde{\psi}(D_x/k)\chi(x)f(s, x) ds \right) \\ &\quad - ik^{2r}\tilde{\psi}(D_x/k)\chi(x)g(t, x), \end{aligned}$$

so

$$(D_t + P_k)v_j = (-1)^j ik^{2r}\tilde{\psi}(D_x/k)\chi(x)f(t, x),$$

where the computation for v_2 follows similarly.

To prove our estimate we need only show

$$\|\chi\tilde{\psi}v_j\|_{L^2([0, T]; L_x^2)} \leq C\|f\|_{L^2([0, T]; L_x^2)}.$$

Taking a Fourier transform in time, we see

$$(\tau + P_k)\hat{v}_j = k^{2r}\tilde{\psi}(D_x/k)\chi(x)(-1)^j\hat{i}(\tau, x).$$

Next, let τ be such that $(P_k + \tau)^{-1}$ exists. Then

$$\begin{aligned}\hat{v}_1 &= k^{2r} \int_{\mathbb{R}} e^{-it\tau} \int_0^t e^{i(s-t)P_k} \tilde{\psi}(D_x/k)\chi(x)f(s, x) ds dt \\ &= k^{2r} \int_{\mathbb{R}} \int_0^t e^{isP_k - it(P_k + \tau)} \tilde{\psi}(D_x/k)\chi(x)f(s, x) ds dt \\ &= k^{2r} \int_{\mathbb{R}} \int_0^t i(P_k + \tau)^{-1} \frac{d}{dt} (e^{isP_k - it(P_k + \tau)} \tilde{\psi}(D_x/k)\chi(x)f(s, x)) ds dt \\ &= k^{2r} i(P_k + \tau)^{-1} \int_{\mathbb{R}} (e^{it\tau}) \tilde{\psi}(D_x/k)\chi(x)f(t, x) dt \\ &= k^{2r} i(P_k + \tau)^{-1} \tilde{\psi}\chi\hat{f}(\tau, x),\end{aligned}$$

and similarly for \hat{v}_2 . Thus for real τ and $\epsilon > 0$,

$$\begin{aligned}\|\chi\tilde{\psi}v_j\|_{L^2([0, T]; L_x^2)} &= \|e^{\epsilon t} e^{-\epsilon t} \chi\tilde{\psi}v_j\|_{L^2([0, T]; L_x^2)} \\ &\leq e^{\epsilon T} \|\chi\tilde{\psi}v_j(\cdot - i\epsilon)\|_{L_{\tau}^2(\mathbb{R}; L_x^2)} \\ &\leq e^{\epsilon T} \|\chi\tilde{\psi}k^{2r}(P_k + \tau - i\epsilon)^{-1}\tilde{\psi}\chi\hat{f}(\tau - i\epsilon, x)\|_{L_{\tau}^2(\mathbb{R}; L_x^2)} \\ &\leq C e^{\epsilon T} \|\chi\tilde{\psi}k^{2r}(P_k + \tau - i\epsilon)^{-1}\tilde{\psi}\chi\|_{L_x^2 \rightarrow L_x^2} \|f\|_{L_{\tau}^2(\mathbb{R}; L_x^2)},\end{aligned}$$

using the fact that $(P_k + \tau - i\epsilon)^{-1}$ exists. Therefore we need to show

$$\|\chi\tilde{\psi}k^{2r}(P_k + \tau - i\epsilon)^{-1}\tilde{\psi}\chi\|_{L_x^2 \rightarrow L_x^2} \leq C.$$

Finally, set $-z = \tau k^{-2}$, $h = |k|^{-1}$, and $V = A^{-2}(x) + h^2 V_1(x)$. Then this becomes

$$\|\chi\tilde{\psi}((hD_x)^2 + V - z \pm i0)^{-1}\tilde{\psi}\chi\|_{L_x^2 \rightarrow L_x^2} \leq Ch^{-2(1-r)}.$$

□

We will prove (2.4) using a gluing argument.

3. SEMICLASSICAL CALCULUS

Because we will be working with h -dependent neighborhoods, we must use a calculus which allows our symbols to depend on h . Generally, we will have symbols of the form $\chi(x/h^\gamma, \xi/h^\beta)$, where $\chi \in C_0^\infty$. Using the rescaling $x \mapsto h^{\frac{\gamma-\beta}{2}}\tilde{x}$, $\xi \mapsto h^{\frac{\beta-\gamma}{2}}\tilde{\xi}$, we can make the rescaling symmetric. We say $a \in S_\delta$ if

$$|\partial^\alpha a| \leq C_\alpha h^{-\delta|\alpha|},$$

so the symbol $\chi(x/h^\gamma, \xi/h^\beta)$ is in the symbol class S_δ for $\delta = (\gamma + \beta)/2$. Some standard results in this calculus are:

Proposition 3.1. *If $a, b \in S_\delta$, then $a^w(x, hD)b^w(x, hD) = c^w(x, hD)$ for $c \in S_\delta$,*

$$c = ab + \frac{h^{1-2\delta}}{2i}\{a, b\} + O_{S_\delta}(h^{2(1-2\delta)}),$$

and

$$[a^w(x, hD), b^w(x, hD)] = \frac{h^{1-2\delta}}{i}\{a, b\}^w(x, hD) + O_{S_\delta}(h^{3(1-2\delta)}).$$

Proposition 3.2. *If $a \in S_\delta$ for $\delta \in (0, 1/2)$, then*

$$\|a^w(x, hD)\|_{L^2 \rightarrow L^2} \leq C \sum_{|\alpha| \leq 2n+1} h^{|\alpha|/2} \|\partial^\alpha a\|_{L^\infty},$$

where C depends only on the dimension.

Proposition 3.3 (Sharp Gårding Inequality). *Let $a \in S_\delta$, $a \geq 0$. Then*

$$\langle a^w(x, hD)u, u \rangle \geq -Ch^{1-2\delta} \|u\|_{L^2}^2.$$

For reasons made clear by these results, we will assume throughout that $0 \leq \delta < 1/2$. See [Zwo12] for further results and exposition.

4. PROPAGATION OF SINGULARITIES FOR h -DEPENDENT SYMBOLS

In this section, we prove a refinement of the standard propagation of singularities lemma for h -dependent symbols. One interpretation of the propagation of singularities theorem is from a control theory point of view. Roughly, it says that if P is a pseudodifferential operator with principal symbol p , and V_1, V_2 are sets in phase space such that V_1 flows into V_2 under the Hamiltonian flow of p , then the L^2 mass of a function u in V_1 is controlled by that in V_2 , modulo a term with Pu along the classical flow. Furthermore, by allowing the region V_2 to grow as a function of h , we obtain constants which are h -dependent in a favorable way. This will be necessary when we apply this estimate an h -dependent number of times and sum the results.

This is a generalization of results in [Chr13, Appendix] and [Chr08]. Our proof uses the same approach (see also Hörmander's original proof [Hör71] and [Tay81]). For a detailed exposition of a similar result, see [Dya11].

Lemma 4.1. *Let $G = g^w(x, hD)$ for $g \in S_\delta^0 \cap C_0^\infty$ real. Let $P = p^w(x, hD)$ for real $p \in S_\delta(m)$ for some order function m , let p_0 be the principal symbol of p , and let $\beta(h) \geq 1$. Then for any k and N ,*

$$\begin{aligned} \frac{\beta(h)}{2h} \|G(P - z)u\|^2 &\geq \frac{h}{2} \langle \alpha^w u, u \rangle - \mathcal{O}(h^{\min(3(1-2\delta), 2)}) \|\tilde{G}u\|^2 \\ &\quad - \mathcal{O}(h^N) \|(\langle hD \rangle \langle x \rangle)^{-k} u\|^2, \end{aligned}$$

where

$$\alpha(x, \xi) = \{p_0, g^2\} - \frac{g^2}{\beta(h)}$$

and

$$\tilde{G} \equiv 1 \text{ on } \text{WF}_h(G),$$

with $\tilde{G} = \tilde{g}^w$ for $\tilde{g} \in C_0^\infty$.

Proof. We have

$$\begin{aligned} \frac{\beta(h)}{2h} \|G(P - z)u\|^2 + \frac{h}{2\beta(h)} \|Gu\|^2 &\geq \text{Im} \langle G(P - z)u, Gu \rangle \\ &= (2i)^{-1} (\langle G(P - z)u, Gu \rangle - \langle Gu, G(P - z)u \rangle) \\ &= (2i)^{-1} (\langle [G, P - z]u, Gu \rangle - \langle Gu, [G, P - z]u \rangle \\ &\quad + \langle (P - z)Gu, Gu \rangle - \langle Gu, (P - z)Gu \rangle) \\ &= \text{Im} \langle [G, P]u, Gu \rangle \\ &= \text{Im} \langle G[G, P]u, u \rangle. \end{aligned}$$

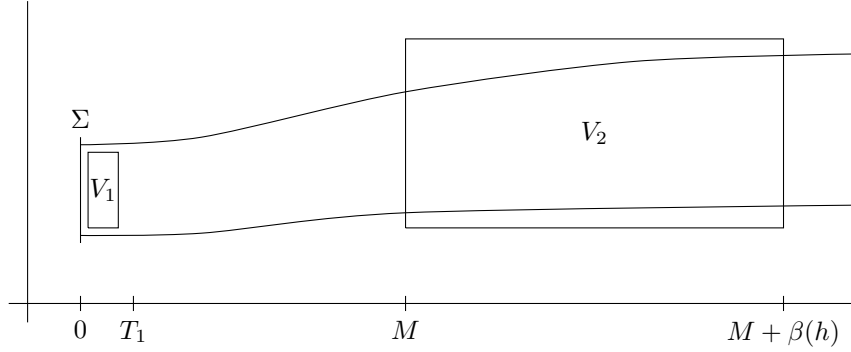


FIGURE 2. The h -dependent neighbourhoods used in the setup of Theorem 2, and some representative flow lines for the H_p flow.

Now,

$$\begin{aligned} g^w[g^w, p^w] &= \frac{h}{i} g^w \{g, p\}^w + h^{3(1-2\delta)} g^w r^w \\ &= \frac{h}{i} (g\{g, p\})^w - \frac{h^2}{2} \{g, \{g, p\}\}^w + h^{3(1-2\delta)} g^w r^w \\ &= \frac{ih}{2} \{p, g^2\}^w - \frac{h^2}{2} \{g, \{g, p\}\}^w + h^{3(1-2\delta)} g^w r^w, \end{aligned}$$

for $r \in S^0$. Since g and p are real, the symbols $\{g, \{g, p\}\}$ and $\{p, g^2\}$ are real, and the operators given by their quantizations are self-adjoint. Thus

$$\operatorname{Im} \langle G[G, P]u, u \rangle = \frac{h}{2} \langle \{p, g^2\}^w u, u \rangle + \mathcal{O}(h^{3(1-2\delta)}) \langle Gr^w u, u \rangle.$$

Next let $\tilde{G} \in C_0^\infty$, with $\tilde{G} \equiv 1$ on $\operatorname{WF}_h G$, $\tilde{G} = \tilde{g}^w$. Then for any N we expand

$$\tilde{G}(Gr^w)\tilde{G} = (\tilde{g}gr\tilde{g})^w + \mathcal{O}_{S^0}(h^{N(1-2\delta)}),$$

where the higher order terms disappear because \tilde{g} is constant on the support of g . Thus on the right hand side we are left with g^w and the error term, so

$$\langle Gr^w u, u \rangle = \langle (Gr^w)\tilde{G}u, \tilde{G}u \rangle + h^{N(1-2\delta)} \langle a^w u, u \rangle,$$

for $a \in \mathcal{S}$. Then

$$\begin{aligned} |\langle a^w u, u \rangle| &= \left| \langle \langle x \rangle^{-k} \langle x \rangle^k a^w u, u \rangle \right| = \left| \langle (\langle hD \rangle \langle x \rangle)^k a^w u, (\langle hD \rangle \langle x \rangle)^{-k} u \rangle \right| \\ &\leq \|(\langle hD \rangle \langle x \rangle)^k a^w u\| \|(\langle hD \rangle \langle x \rangle)^{-k} u\|. \end{aligned}$$

Since $(\langle hD \rangle \langle x \rangle)^k a^w = c^w$ for $c \in \mathcal{S}$, we can then bound this term by $\|(\langle hD \rangle \langle x \rangle)^{-k} u\|^2$ for any k . Putting everything together, applying Cauchy-Schwarz where necessary, and rearranging, we obtain the result with p instead of p_0 .

To replace p with p_0 we write $p = p_0 + h\tilde{p}$. Plugging this into α we obtain the term $h^2 \langle \{\tilde{p}, g^2\} u, u \rangle$, which we can absorb into the $\|\tilde{G}u\|^2$ term, taking the worse power of h . \square

Theorem 2. *Let $V_1, V_2 \Subset T^*M$. Let Σ be a non-characteristic hypersurface, and $T_1 > 0$ such that*

$$(4.1) \quad V_1 \subset \bigcup_{0 \leq t \leq T_1} \exp(tH_{p_0})(\Sigma).$$

Let $M, \beta(h) \geq 1$ be such that

$$(4.2) \quad \bigcup_{M \leq t \leq M+T_1+\beta(h)} \exp(tH_{p_0})(\Sigma) \subset V_2.$$

Let $A = a^w(x, hD)$, $a \in S_0^0$, $a \equiv 1$ on V_2 . Let $B = b^w(x, hD)$ for $b \in S_\delta^0$, $WF_h(B) \subset V_1$. Then for any N and k , there exists a constant $C > 0$, independent of $M, \beta(h)$, and h such that for any $z \in \mathbb{R}$,

$$\begin{aligned} \|Bu\| &\leq C\sqrt{\beta(h)}h^{-1}\|B\|_{L^2 \rightarrow L^2}\|(P-z)u\| + C\frac{\|B\|_{L^2 \rightarrow L^2}}{\sqrt{\beta(h)}}\|Au\| \\ &\quad + Ch^{\min(1/2-\delta, 1-3\delta)}\|\tilde{B}u\| + Ch^N\|(\langle hD \rangle \langle x \rangle)^{-k}u\|, \end{aligned}$$

for

$$\tilde{B} \equiv 1 \text{ on } \bigcup_{0 \leq t \leq M+\beta(h)} \exp(tH_{p_0})(WF_h B),$$

and $WF_h(B)$ is compact.

Remark 4.2. The sets V_1 and V_2 may depend on h . Indeed, the h -dependence of V_2 is more or less explicitly given by $\beta(h)$ and H_p . For typical applications, we will have V_1 shrinking with h .

Remark 4.3. The idea of conditions (4.1) and (4.2) is that V_1 flows into V_2 and remains inside that region for a time of length $\beta(h)$. Indeed, we have

$$\bigcup_{M \leq t \leq M+\beta(h)} \exp(tH_{p_0})(V_1) \subset \bigcup_{M \leq t \leq M+T_1+\beta(h)} \exp(tH_{p_0})(\Sigma) \subset V_2$$

Remark 4.4. Note that we may apply the lemma repeatedly to the error term involving \tilde{B} in order to obtain better powers of h .

Proof. Let $\tilde{\varphi}_0 \in C^\infty(\mathbb{R})$ be such that

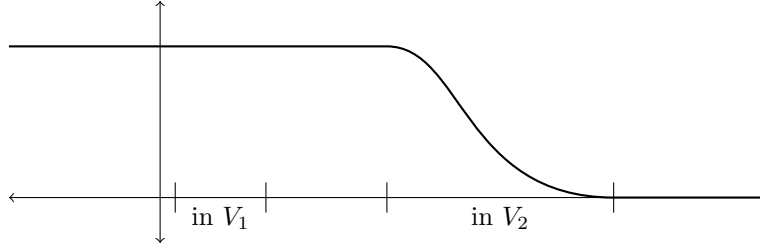
$$\begin{aligned} \tilde{\varphi}_0(t) &= \tilde{\varphi}_1^2 \in C^\infty \\ \tilde{\varphi}_0(t) &\equiv 1 \text{ for } t < M, \\ \text{supp } \tilde{\varphi}_0'(t) &\subset (M, M+T_1+\beta(h)), \\ \tilde{\varphi}_0'(t) &\leq 0, \text{ and} \\ -\tilde{\varphi}_0'(t) &\leq \frac{2}{\beta(h)}. \end{aligned}$$

Let $(x_0, \xi_0) \in \Sigma$. We define $\varphi_0(\exp(tH_{p_0})(x_0, \xi_0)) = \tilde{\varphi}_0(t)$. This defines φ_0 on the flowout of Σ . Its value elsewhere will not matter. Note that

$$\tilde{\varphi}_0'(t) = H_{p_0}\varphi_0.$$

Thus φ_0 is such that

$$\varphi_0 = \varphi_1^2 \in C^\infty \quad \text{and} \quad \varphi_0 \equiv 1 \text{ on } V_1.$$

FIGURE 3. φ_0 along an integral curve of H_{p_0} .

Choose $\chi_0 \in C_0^\infty$ with $\chi_0 \equiv 1$ on V_1 , with $\text{supp } \chi_0 \subset \{\varphi_0 = 1\}$. We can choose χ in an h independent way because M does not depend on h . Note however that some h -dependence would be fine (as long as $\chi_0 \in S_\delta^0$.)

Let $\varphi \in C_0^\infty$, $0 \leq \varphi \leq 1$ be such that $\varphi(p_0(x, \xi)) \equiv 1$ on $\text{supp } \chi_0$ and φ has support near the energy levels of V_1 . In particular we require

$$\text{supp } \varphi(p_0) \subset \bigcup_{-\infty < t < \infty} \exp(tH_{p_0})(\Sigma).$$

This allows us to deal with φ_0 only on the flowout of Σ .

$$\varphi(p_0)H_{p_0}\varphi_0 \leq 0 \quad \text{and} \quad -\varphi(p_0)H_{p_0}\varphi_0 \leq \frac{2}{\beta(h)}.$$

Let

$$\tilde{\Sigma} = \bigcup_{0 \leq t \leq M+T_1+\beta(h)} \exp(tH_{p_0})(\Sigma).$$

Let $f \in C_0^\infty(\Sigma)$ such that

$$\bigcup_{0 \leq t \leq T_1} \exp(tH_{p_0})(\text{supp } f) \subset V_1.$$

Let $q, a_0 \in C^\infty(T^*\mathbb{R})$ be the solutions to

$$H_{p_0}q = \chi_0, \quad q|_\Sigma = f,$$

$$H_{p_0}a_0 = 1, \quad a_0|_\Sigma = 0.$$

These solutions are obtained by integrating starting at Σ along integral curves, so

$$\begin{aligned} 0 &\leq a_0 \leq M + T_1 + \beta(h) \quad \text{on } \tilde{\Sigma}, \\ 1 &\leq q \leq 1 + T_1 \quad \text{on } V_1, \end{aligned}$$

and $q \in S_\delta^0$.

Let $\psi \in C^\infty$ be such that $\psi \equiv 0$ on $\{\exp(tH_{p_0})(\Sigma) : t < -L\}$ for some $L > 0$, $\psi \equiv 1$ on $\{\exp(tH_{p_0}) : t > -L + 1\}$, and $H_{p_0}\psi \geq 0$.

Now define

$$g^2 = \varphi^2(p_0)\varphi_0(x, \xi)(\psi q)^2 \exp(a_0/\beta(h)).$$

Note that $\varphi(p_0) = 0$ if we are away from the energy levels containing V_1 and V_2 , $\varphi_0 = 0$ if we flow beyond V_2 , and $\psi = 0$ if we flow backwards in time far enough. Thus g is compactly supported.

Let

$$\begin{aligned}
\alpha(x, \xi) &= \{p_0, g^2\} - \frac{1}{\beta(h)} g^2 \\
&= \{p_0, \varphi_0\} \varphi^2(p) (\psi q)^2 \exp(a_0/\beta(h)) + 2\varphi_0 \varphi^2(p) q \{p_0, \psi q\} \exp(a_0/\beta(h)) \\
&\quad + \frac{1}{\beta(h)} \varphi_0 \varphi^2(p_0) (q\psi)^2 \{p_0, a_0\} \exp(a_0/\beta(h)) - \frac{1}{\beta(h)} g^2 \\
&= (H_{p_0} \varphi_0) \varphi^2(p_0) (q\psi)^2 \exp(a_0/\beta(h)) + 2\varphi_0 \varphi^2(p_0) q (H_{p_0}(\psi q)) \exp(a_0/\beta(h))
\end{aligned}$$

By Lemma 4.1,

$$\begin{aligned}
\frac{\beta(h)}{h} \|G(P - z)u\|^2 &\geq h \langle \text{Op}^w((H_{p_0} \varphi_0) \varphi^2(p_0) (q\psi)^2 \exp(a_0/\beta(h))) u, u \rangle \\
&\quad + h \langle \text{Op}^w(\varphi_0 \varphi^2(p_0) q (H_{p_0}(\psi q)) \exp(a_0/\beta(h))) u, u \rangle \\
&\quad - \mathcal{O}(h^{\min(3(1-2\delta), 2)}) \|\tilde{B}u\|^2 - \mathcal{O}(h^N) \|(\langle hD \rangle \langle x \rangle)^{-k} u\|^2
\end{aligned}$$

Rearranging,

(4.3)

$$\begin{aligned}
\langle \text{Op}^w(\varphi_0(x) \varphi^2(p_0) (H_{p_0}(\psi q)) \exp(a_0/\beta(h))) u, u \rangle &\leq \frac{\beta(h)}{h^2} \|G(P - z)u\|^2 \\
&\quad - \langle \text{Op}^w((H_{p_0} \varphi_0) \varphi^2(p_0) (q\psi)^2 \exp(a_0/\beta(h))) u, u \rangle \\
&\quad + \mathcal{O}(h^{\min(2-6\delta, 1)}) \|\tilde{B}u\|^2 + \mathcal{O}(h^N) \|(\langle hD \rangle \langle x \rangle)^{-k} u\|^2
\end{aligned}$$

We have

$$\begin{aligned}
\varphi_0 \varphi^2(p_0) q (H_{p_0}(\psi q)) \exp(a_0/\beta(h)) &\geq H_{p_0}(\psi q) \\
&= \psi \chi_0 + (H_p \psi) \chi_0 \\
&\geq \chi_0(x, \xi),
\end{aligned}$$

and

$$\begin{aligned}
&-(H_{p_0} \varphi_0) \varphi^2(p_0) (\psi q)^2 \exp(a_0/\beta(h)) \\
&\leq -\exp((M + T_1 + \beta(h))/\beta(h)) (H_{p_0} \varphi_0) \varphi^2(p_0) (q\psi)^2 \\
&\leq -C \exp(M + T_1) (H_{p_0} \psi_0) \psi^2(p_0) (q\psi)^2
\end{aligned}$$

Using these estimate to apply the sharp Gårding inequality to (4.3), we obtain

$$\begin{aligned}
\langle \text{Op}^w(\chi_0(x, \xi)) u, u \rangle &\leq \frac{\beta(h)}{h^2} \|G(P - z)u\|^2 - M \langle \text{Op}^w(H_{p_0} \varphi_0(x) (\psi q)^2) u, u \rangle \\
&\quad + \mathcal{O}(h^\alpha) \|\tilde{B}u\|^2 + \mathcal{O}(h^N) \|(\langle hD \rangle \langle x \rangle)^{-k} u\|^2,
\end{aligned}$$

where $\alpha = \min(1-2\delta, 2-6\delta)$. Note that $\chi_0 \equiv 1$ on V_1 , $\text{supp}(H_{p_0} \varphi_0) \varphi^2(p) (\psi q)^2 \subset V_2$, and $-(H_{p_0} \varphi_0) \varphi^2(p) (\psi q)^2 \leq 2/\beta(h)$, so

$$\begin{aligned}
\|Bu\|^2 &\leq \frac{\beta(h)}{h^2} \|B\|^2 \|G(P - z)u\|^2 - \frac{C}{\beta(h)} \|B\|^2 \|Au\|^2 \\
&\quad + \mathcal{O}(h^\alpha) \|B\|^2 \|\tilde{B}u\|^2 + \mathcal{O}(h^N) \|(\langle hD \rangle \langle x \rangle)^{-k} u\|^2.
\end{aligned}$$

Taking the square root obtains the inequality. \square

5. GLUING WITH AN h -DEPENDENT NUMBER OF h -DEPENDENT REGIONS

Let $P = p^w(x, hD)$ for $p(x, \xi) = |\xi|^2 + V(x)$, where $V(x)$ is a short range perturbation of the inverse square potential x^{-2} . Then the trapped set of H_p is contained in a compact set K . Outside K we can replace p with a globally non-trapping symbol \tilde{p} such that we have a $\mathcal{O}(h)$ bound on the cutoff resolvent of $\tilde{P} = \tilde{p}^w(x, hD)$. Let $\Gamma \in C_0^\infty(\mathbb{R}^n)$ be such that $\Gamma \equiv 1$ on K . We decompose Γ into a sum of microlocal cutoffs Γ_j such that on the support of each Γ_j we have microlocal estimates established through other means. We are able to glue these estimates together using the propagation of singularities theorem to control the commutator terms supported in K by terms supported in the non-trapping region.

Theorem 3. *Let $P = p^w(x, hD)$ for $p \in S_\delta^m$ as above and suppose the trapped set of H_{p_0} is contained in some compact set $K_1 \subset K = \{(x, \xi) \in \mathbb{R}^{2n} : |(x, \xi)| \leq M\}$, with $\mathbb{R}^{2n} \setminus K$ flowing to infinity under H_{p_0} . Suppose also that p is independent of h outside of K . Let $\Gamma \equiv 1$ on K with Γ independent of h and suppose that we can write*

$$\Gamma = \sum_{j=1}^{N(h)} \Gamma_j,$$

and for each $\Gamma_j = \gamma_j^w(x, hD)$, we have $\gamma_j \equiv 1$ in a neighborhood of a critical point of H_p . Suppose also that there exist C, k such that $N(h) \leq Ch^{-k}$, and that we have estimates of the form

$$\|\Gamma_j u\| \leq \frac{\alpha_j(h)}{h} \|(P - z)\Gamma_j u\|.$$

Furthermore assume $\gamma_j \in S_\delta^0$ for $\delta < 1/3$ and that

$$(5.1) \quad \|\partial^\alpha \gamma_j\| \leq C_\alpha h^{-\delta|\alpha|},$$

with each C_α independent of j .

We also require a dynamical condition on our microlocal cutoffs: With $V_j \subset \mathbb{R}^{2n}$ as the region on which γ_j is non-constant, we must have that V_j eventually flows outside of K .

Then

$$\|\rho_s(P - z)u\|^2 \geq C \frac{h^2 h^{4\delta}}{N(h)^4} \min\{N(h)h^{-4\delta}, \alpha_1^{-2}(h), \dots, \alpha_N^{-2}(h)\} \|\rho_{-s}u\|^2,$$

where $\rho_s \in C^\infty(M)$, $\rho_s > 0$, $\rho_s \equiv 1$ on a neighborhood of $K_2 = \{x : |x| \leq 2M\}$, and $\rho_s = \langle x \rangle^s$ for x large, and $s > 1$.

Remark 5.1. The dynamical condition is to ensure that we may apply the propagation of singularities theorem.

Remark 5.2. If $N(h)$ is independent of h then the bound (5.1) follows automatically from the symbol classes of the γ_j . Otherwise, it is there to prevent the case where γ_j has derivatives which grows with j .

Remark 5.3. The requirement on δ comes from the propagation of singularities theorem. We require $\delta < 1/3$ in order to get an error term which has a positive power of h .

Proof. We have

$$\begin{aligned} |\langle \rho_s(1 - \Gamma)(P - z)u, \Gamma(P - z)u \rangle| &\leq \|\rho_s(1 - \Gamma)(P - z)u\| \|\Gamma(P - z)u\| \\ &\leq C \|\rho_s(P - z)u\|^2, \end{aligned}$$

so

$$\begin{aligned} \|\rho_s(P - z)u\|^2 &\geq \|\rho_s(1 - \Gamma)(P - z)u\|^2 + \|\Gamma(P - z)u\|^2 \\ &\quad - 2|\langle \rho_s(1 - \Gamma)(P - z)u, \Gamma(P - z)u \rangle| \\ &\geq \|\rho_s(1 - \Gamma)(P - z)u\|^2 + \|\Gamma(P - z)u\|^2 - 2C \|\rho_s(P - z)u\|^2, \end{aligned}$$

so

$$C \|\rho_s(P - z)u\|^2 \geq \|\rho_s(1 - \Gamma)(P - z)u\|^2 + \|\Gamma(P - z)u\|^2.$$

Similarly, using 5.1, we have

$$\begin{aligned} (5.2) \quad |\langle \Gamma_j(P - z)u, \Gamma_k(P - z)u \rangle| &\leq \|\Gamma_j(P - z)u\| \|\Gamma_k(P - z)u\| \\ &\leq C \|(P - z)u\|^2 \\ &\leq C \|\rho_s(P - z)u\|^2, \end{aligned}$$

so we can similarly decompose the $\Gamma(P - z)u$ term. When we decompose, we will get $N(h)(N(h) - 1)$ terms which we bound using (5.2) and move to the left hand side, so we obtain

$$\tilde{N}(h) \|\rho_s(P - z)u\|^2 \geq C(\|\rho_s(1 - \Gamma)(P - z)u\|^2 + \sum_{j=1}^{N(h)} \|\Gamma_j(P - z)u\|^2),$$

where $\tilde{N}(h) = C(N(h)^2 - N(h)) + C \leq CN(h)^2$. Next we multiply through by an h -dependent constant $A(h) \geq 1$ to obtain

$$\begin{aligned} \tilde{N}(h)A(h) \|\rho_s(P - z)u\|^2 &\geq CA(h)(\|\rho_s(1 - \Gamma)(P - z)u\|^2 + \sum_{j=1}^{N(h)} \|\Gamma_j(P - z)u\|^2) \\ &\geq C(A(h) \|\rho_s(1 - \Gamma)(P - z)u\|^2 + \sum_{j=1}^{N(h)} \|\Gamma_j(P - z)u\|^2). \end{aligned}$$

We then have

$$\begin{aligned} \|\rho_s \Gamma_j(P - z)u\|^2 &= \|\rho_s(P - z) \Gamma_j u + [P, \Gamma_j]u\|^2 \\ &= \|\rho_s(P - z) \Gamma_j u\|^2 + \|[P, \Gamma_j]u\|^2 \\ &\quad + 2 \operatorname{Re} \langle \rho_s(P - z) \Gamma_j u, [P, \Gamma_j]u \rangle \\ &\geq \|\rho_s(P - z) \Gamma_j u\|^2 + \|[P, \Gamma_j]u\|^2 \\ &\quad - 2 \|\rho_s(P - z) \Gamma_j u\| \|[P, \Gamma_j]u\|, \end{aligned}$$

Combining this with the above

$$\begin{aligned} CN(h)^2 A(h) \|\rho_s(P - z)u\|^2 &\geq CA(h) \|\rho_s(1 - \Gamma)(P - z)u\|^2 \\ &\quad + \sum_{j=1}^{N(h)} (\|\rho_s(P - z) \Gamma_j u\|^2 + \|[P, \Gamma_j]u\|^2 \\ &\quad - 2 \|\rho_s(P - z) \Gamma_j u\| \|[P, \Gamma_j]u\|). \end{aligned}$$

Then applying the inequality $2ab \leq a^2 + b^2$, we have for $\eta > 0$

$$\begin{aligned} CN(h)^2 A(h) \|\rho_s(P-z)u\|^2 &\geq A(h) \|\rho_s(1-\Gamma)(P-z)u\|^2 \\ &\quad + \sum_{j=1}^{N(h)} (\|\rho_s(P-z)\Gamma_j u\|^2 + \|[P, \Gamma_j]u\|^2 \\ &\quad - \eta \|\rho_s(P-z)\Gamma_j u\|^2 - \eta^{-1} \|[P, \Gamma_j]u\|^2). \end{aligned}$$

Then by taking η small but fixed, we have

$$\begin{aligned} (5.3) \quad CN(h)^2 A(h) \|\rho_s(P-z)u\|^2 &\geq CA(h) (\|\rho_s(1-\Gamma)(P-z)u\|^2 \\ &\quad + \sum_{j=1}^{N(h)} (C \|\rho_s(P-z)\Gamma_j u\|^2 - C \|[P, \Gamma_j]u\|^2)). \end{aligned}$$

Let $\psi \equiv 1$ on a set which grows like $\beta(h)$. Then

$$\psi \leq C\beta(h)^s \rho_s.$$

We claim that for some rapidly decaying $\tilde{\chi}$, we have

$$\begin{aligned} (5.4) \quad \|\rho_s(1-\Gamma)(P-z)u\|^2 &\geq Ch^2 \beta^{-2s}(h) \|\psi(1-\Gamma)u\|^2 + Ch^2 \|\rho_{-s}(1-\Gamma)u\|^2 \\ &\quad - C \|(P-z)u\|^2 - Ch^6 \|\tilde{\chi}u\|^2. \end{aligned}$$

Applying this, we obtain

$$\begin{aligned} (5.5) \quad CN(h)^2 A(h) \|\rho_s(P-z)u\|^2 &\geq CA(h) h^2 \beta(h)^{-2s} \|\psi(1-\Gamma)u\|^2 \\ &\quad + \sum_{j=1}^{N(h)} (C \|\rho_s(P-z)\Gamma_j u\|^2 - C \|[P, \Gamma_j]u\|^2) \\ &\quad + CA(h) h^2 \|\rho_{-s}(1-\Gamma)u\|^2 - Ch^6 \|\tilde{\chi}u\|^2, \end{aligned}$$

where we have moved the $\|(P-z)u\|^2$ term to the left hand side.

Now we will apply Theorem 2 to the commutator terms $[P, \Gamma_\star]$. Note that the commutator $[P, \Gamma_\star]$ is $\mathcal{O}(h)$, and has wavefront set contained in the region on which Γ_\star is non-constant. Write $[P, \Gamma_\star] = h^{1-2\delta} B_\star$, with $B_\star \in S_\delta^0$, and in particular $H_p \neq 0$ on $\text{WF}_h(B_\star)$.

Let $A = \psi(1-\Gamma)$, where $\psi \equiv 1$ on a set which grows like $\beta(h)$. Then

$$(5.6) \quad \psi \leq C \langle \beta(h) \rangle^s \langle x \rangle^{-s}.$$

We get, for any k ,

$$\begin{aligned} \|[P, \Gamma_\star]u\| &\leq C \sqrt{\beta(h)} h^{-2\delta} \|(P-z)u\| + C \frac{h^{1-2\delta}}{\sqrt{\beta(h)}} \|\psi(1-\Gamma)u\| \\ &\quad + Ch^{1-2\delta+\alpha} \|\tilde{\chi}u\| + Ch^6 \|(\langle hD \rangle \langle x \rangle)^{-k} u\|, \end{aligned}$$

where $\alpha > 0$ because $\delta < 1/3$, and $\beta(h)$ will be chosen later. Here $\tilde{\chi} \equiv 1$ is compactly supported. We require a better error term than $h^{1-2\delta+\alpha}$, so we apply the propagation of singularities theorem a finite number of times to the $\tilde{\chi}u$ term, with the same A . Each time we gain an h^α and so we obtain

$$\begin{aligned} \|[P, \Gamma_\star]u\| &\leq C \sqrt{\beta(h)} h^{-2\delta} \|(P-z)u\| + C \frac{h^{1-2\delta}}{\sqrt{\beta(h)}} \|\psi(1-\Gamma)u\| \\ &\quad + Ch^6 \|\tilde{\chi}u\| + Ch^6 \|(\langle hD \rangle \langle x \rangle)^{-k} u\|, \end{aligned}$$

where now $\tilde{\chi}$ has larger support. We may then combine the two error terms and assume from now on that $\tilde{\chi}$ decays like $(\langle hD \rangle \langle x \rangle)^{-k}$.

Plugging into (5.5), we get

$$\begin{aligned} CN(h)^2 A(h) \|\rho_s(P-z)u\|^2 &\geq CA(h)h^2\beta(h)^{-2s} \|\psi(1-\Gamma)u\| \\ &\quad + CA(h)h^2 \|\rho_{-s}(1-\Gamma)u\|^2 + \sum_{j=1}^{N(h)} C \|\rho_s(P-z)\Gamma_j u\|^2 \\ &\quad - CN(h)\beta(h)h^{-4\delta} \|(P-z)u\|^2 - CN(h) \frac{h^{2(1-2\delta)}}{\beta(h)} \|\psi(1-\Gamma)u\|^2 \\ &\quad - Ch^6 \sum_{j=0}^{N(h)} \|\tilde{\chi}u\|^2. \end{aligned}$$

Moving the $\|(P-z)u\|^2$ term to the left hand side, and applying the black box estimates, we obtain

$$\begin{aligned} (CN(h)^2 A(h) + CN(h)\beta(h)h^{-4\delta}) \|\rho_s(P-z)u\|^2 &\geq \\ &\quad CA(h)\beta(h)^{-2s} h^2 \|\psi(1-\Gamma)u\|^2 + CA(h)h^2 \|\rho_{-s}(1-\Gamma)u\|^2 \\ &\quad + \sum_{j=1}^{N(h)} C \frac{h^2}{\alpha_j^2(h)} \|\Gamma_j u\|^2 - CN(h) \frac{h^{2(1-2\delta)}}{\beta(h)} \|\psi(1-\Gamma)u\|^2 \\ &\quad - CN(h)h^6 \|\tilde{\chi}u\|^2. \end{aligned}$$

Set

$$A(h) = MN(h)\beta(h)^{2s-1}h^{-4\delta}, \quad \beta(h) = N(h)^{1/(2s-1)},$$

for some large M . Note that this allows us to absorb the negative term involving $\|\psi(1-\Gamma)u\|^2$ as long as M is large enough.

The above is then

$$\begin{aligned} (CN(h)^4 h^{-4\delta} + CN(h)^{2s/(2s-1)} h^{-4\delta}) \|\rho_s(P-z)u\|^2 &\geq \\ &\geq CN(h)^{(2s-2)/(2s-1)} h^{-4\delta} h^2 \|\psi(1-\Gamma)u\|^2 + \sum_{j=1}^{N(h)} C \frac{h^2}{\alpha_j^2(h)} \|\Gamma_j u\|^2 \\ &\quad - CN(h)h^6 \|\tilde{\chi}u\|^2 + CN(h)h^{2-4\delta} \|\rho_{-s}(1-\Gamma)u\|^2. \end{aligned}$$

For the left hand side, the term involving $N(h)^4$ is larger as long as $s \geq 2/3$. As long as $s \geq 1$ we have

$$N(h)^{(2s-2)/(2s-1)} \geq 1,$$

so

$$\begin{aligned} CN(h)^4 h^{-4\delta} \|\rho_s(P-z)u\|^2 &\geq Ch^{2-4\delta} \|\psi(1-\Gamma)u\|^2 + \sum_{j=1}^{N(h)} C \frac{h^2}{\alpha_j^2(h)} \|\Gamma_j u\|^2 \\ &\quad + Ch^{2-4\delta} \|\rho_{-s}(1-\Gamma)u\|^2 - CN(h)h^6 \|\tilde{\chi}u\|^2. \end{aligned}$$

We may drop the first term because it is positive. Let

$$\alpha(h) = \min\{N(h)h^{-4\delta}, \alpha_1^{-2}(h), \dots, \alpha_N^{-2}(h)\}.$$

Taking the worst estimate and summing over the partition of unity,

$$CN(h)^4 h^{-4\delta} \|\rho_s(P-z)u\|^2 \geq Ch^2 \alpha(h) \|\rho_{-s}u\|^2 - CN(h)h^6 \|\tilde{\chi}u\|^2.$$

Since $N(h)h^6 \leq N(h)h^{-4\delta}$, and $\tilde{\chi}$ decays more quickly than ρ_{-s} , we may absorb the final negative term to obtain

$$CN(h)^4 h^{-4\delta} \|\rho_s(P-z)u\|^2 \geq Ch^2 \alpha(h) \|\rho_{-s}u\|^2,$$

or

$$\|\rho_s(P-z)u\|^2 \geq \frac{Ch^2 h^{4\delta} \alpha(h)}{CN(h)^4} \|\rho_{-s}u\|^2.$$

□

We prove our claim below.

Proof of 5.4. As before we have

$$\|\rho_s(1-\Gamma)(P-z)u\|^2 \geq C(\|\rho_s(P-z)(1-\Gamma)u\|^2 - \|[P, \Gamma]u\|^2).$$

Since $(1-\Gamma)$ is supported in the non-trapping region, we obtain

$$\begin{aligned} \|\rho_s(P-z)(1-\Gamma)u\|^2 &\geq Ch^2 \|\rho_{-s}(1-\Gamma)u\|^2 \\ &\geq Ch^2 \|\rho_{-s}(1-\Gamma)u\|^2 + Ch^2 \|\rho_{-s}(1-\Gamma)u\|^2 \\ &\geq Ch^2 \beta^{-2s}(h) \|\psi(1-\Gamma)u\|^2 + Ch^2 \|\rho_{-s}(1-\Gamma)u\|^2. \end{aligned}$$

We can not apply propagation of singularities directly to the commutator term because it has non-compact wavefront set (in the ξ direction). Let φ be a function of p with support only at low energies. We then split up the commutator term:

$$\|[P, \Gamma]u\|^2 \leq C(\|[P, \Gamma]\varphi u\|^2 + \|[P, \Gamma](1-\varphi)u\|^2).$$

Note that $[P, \Gamma] = hB$ for some pseudodifferential operator B . We may apply propagation of singularities (several times) to $B\varphi$, with $A = \rho_{-s}(1-\Gamma)$ to obtain

$$\|[P, \Gamma]\varphi u\|^2 \leq C\|(P-z)u\|^2 + Ch^2 \|\rho_{-s}(1-\Gamma)u\|^2 + Ch^N \|\tilde{\chi}u\|^2,$$

for some rapidly decaying $\tilde{\chi}$ and any N . As long as the region on which $\rho_{-s} \equiv 1$ is large enough, the constant in front of the $\|\rho_{-s}(1-\Gamma)u\|^2$ term will be small enough to allow us to absorb this term later.

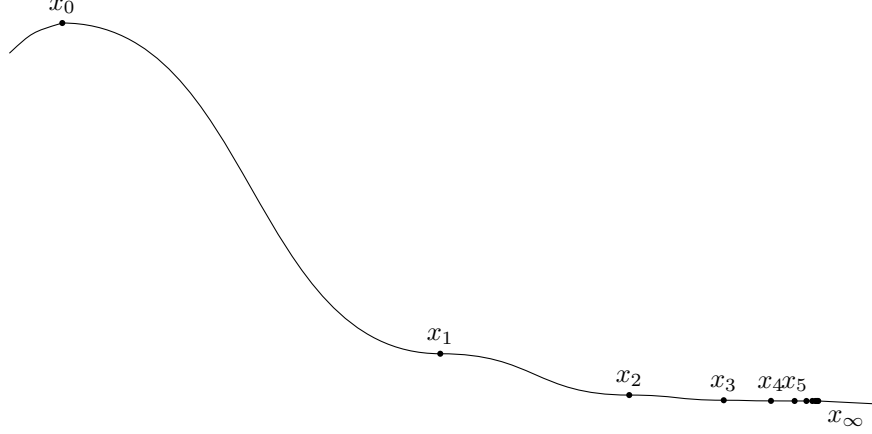
Let $\chi \in C^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ on $\text{supp } \nabla \Gamma$, and supported on a slightly larger region. We have

$$\begin{aligned} \|[P, \Gamma](1-\varphi)u\| &\leq \|\chi(P-z)(1-\varphi)u\| \\ &\leq \|\chi(1-\varphi)(P-z)u\| + \|\chi[P, \varphi]u\| \\ &\leq \|(P-z)u\| + Ch^3 \|Tu\|, \end{aligned}$$

where T is an L^2 bounded operator with $\text{WF}_h(T)$ compact. This allows us to combine it with the term involving $\tilde{\chi}$, so we obtain

$$\begin{aligned} \|\rho_s(1-\Gamma)(P-z)u\|^2 &\geq Ch^2 \beta^{-2s}(h) \|\psi(1-\Gamma)u\|^2 + Ch^2 \|\rho_{-s}(1-\Gamma)u\|^2 \\ &\quad - C\|(P-z)u\|^2 - Ch^6 \|\tilde{\chi}u\|^2. \end{aligned}$$

□

FIGURE 4. The potential V_0 .

6. UNSTABLE TRAPPING WITH INFINITELY MANY CONNECTED COMPONENT

Recall that the potential we are interested in is $V_0(x) = A^{-2}(x)$, with critical points at $x = 0$ and $x = x_n$ for $x_n = 1 - 2^{-n}$. The critical point at $x = 0$ is a non-degenerate maximum while the critical points at x_n are inflection points of order m . That is, at each x_n , we have a Taylor expansion

$$V_0(x) = \alpha_n - \beta_n(x - x_n)^m + \mathcal{O}(|x - x_n|^{m+1}).$$

Here $\{\alpha_n\}$ is a decreasing sequence of critical values.

We denote $d_n = x_{n+1} - x_n$. The point is that, if V_0 is a sufficiently nice function (necessarily non-analytic), then the order of the critical point at x_n dictates a relationship between the d_n s and the α_n s.

Note that the derivatives of V_0 are uniformly bounded on a compact set containing all the critical points of V_0 , so the error in the Taylor expansion at each point is uniform in n . Let $x_\infty = 1$, $x_n = 1 - 2^{-n}$, so that

$$\begin{aligned} d_n &= x_{n+1} - x_n \\ &= (1 - 2^{-n-1}) - (1 - 2^{-n}) \\ &= 2^{-n}(1 - 2^{-1}) \\ &= 2^{-n-1}. \end{aligned}$$

For our sequence of critical values, let $\alpha_n = 1 + 2^{-mn}$, and assume $\beta_n = 1$ for each n . That is, near each x_n , we have

$$V_0(x) = (1 + 2^{-mn}) - (x - (1 - 2^{-n}))^m + \mathcal{O}(|x - (1 - 2^{-n})|^{m+1}),$$

with the error uniform in n . Since V_0 is well approximated by an m -th degree polynomial near each critical point, we need to insist that the change in height of V_0 is sufficiently large. That is, on $[x_n, x_{n+1}]$, we must have

$$\alpha_n - d_n^m \geq \alpha_{n+1},$$

or

$$\alpha_n - \alpha_{n+1} \geq d_n^m.$$

Plugging in our assumed values for this example, we have

$$2^{-mn} - 2^{-mn-m} \geq 2^{-mn-m}.$$

Rearranging, this inequality reads

$$(1 - 2^{-m})2^{-mn} \geq 2^{-m}2^{-mn}.$$

Since $m \geq 3$, this gives us a little wiggle room to ensure that V_0 is \mathcal{C}^∞ under these conditions.

Our strategy for understanding the spectral theory of $-h^2\partial_x^2 + V(x)$ is to replace V_0 with another potential with *finitely* many inflection points, which is equal to V_0 at these points. We will have to do this in an h -dependent fashion. We observe that the function

$$\tilde{V}(x) = 1 - (x - 1)^m$$

has the property that

$$\begin{aligned} \tilde{V}(x_n) &= 1 - ((1 - 2^{-n}) - 1)^m \\ &= 1 + 2^{-mn} \\ &= \alpha_n, \end{aligned}$$

so that \tilde{V} interpolates between the critical values of V_0 without itself having critical points other than at $x = 1$. We will need the difference between V_0 and \tilde{V} to be smaller than the spectral estimate obtained by gluing. In particular, we will need $|V_0(x) - \tilde{V}(x)| \leq ch^{\frac{2m^2}{(m-1)(m+2)} + \epsilon}$ for some small constant c . If $x_n \leq x \leq x_{n+1}$, we have

$$V_0(x_{n+1}) \leq V_0(x) \leq V_0(x_n),$$

or $\alpha_{n+1} \leq V_0(x) \leq \alpha_n$. We have also $\alpha_{n+1} \leq \tilde{V}(x) \leq \alpha_n$ in the same range of x values, since \tilde{V} was designed to interpolate. That means for $x_n \leq x \leq x_{n+1}$, we have

$$\begin{aligned} |V_0(x) - \tilde{V}(x)| &\leq (1 + 2^{-mn}) - (1 + 2^{-mn-m}) \\ &\leq (1 - 2^{-m})2^{-3n}. \end{aligned}$$

Hence $|V_0 - \tilde{V}| \leq ch^{\frac{2m^2}{(m-1)(m+2)} + \epsilon}$ if

$$(1 - 2^{-m})2^{-3n} \leq ch^{\frac{2m^2}{(m-1)(m+2)} + \epsilon},$$

or

$$2^{-n} \leq c'h^{\frac{2m}{(m-1)(m+2)} + \epsilon/m}.$$

for some small $c' > 0$. This tells us that the number of critical points we need to consider is $N = \mathcal{O}(\log(1/h))$, and that we will need to replace V_0 with \tilde{V} where the distance between critical points is smaller than $2^{-n} \leq c'h^{2m/(m-1)(m+2) + \epsilon/m}$. But since $2^{-n} = d_{n-1}$, this tells us that we will have to cutoff V_0 on scales $h^{2m/(m-1)(m+2) + \epsilon/m}$ and consider $\log(1/h)$ terms.

The Hamiltonian flow generated by our symbol p is given by

$$\begin{cases} \dot{x} = 2\xi, \\ \dot{\xi} = -V'_0(x), \end{cases}$$

Let $\xi_0 > 0$, $x_0 \geq 0$ and consider the flow starting at this point. For $x \geq 0$ we have $-V'_0(x) \geq 0$. Thus

$$\xi(t) \geq \xi_0.$$

Therefore $\dot{x} \geq 2\xi_0$, so

$$x(t) \geq 2\xi_0 t + x_0 \geq 2\xi_0 t.$$

A similar argument works as long as $\xi_0 \neq 0$ (though in some of the cases we must flow backwards in time, but this is fine.)

Now, if $\xi_0 = 0$ and x_0 is not a critical point of V_0 , then after some time t_0 , we have $\xi(t_0) \neq 0$. Continuing the flow from $x(t_0)$, $\xi(t_0)$, we are in the case of the above paragraph, and so eventually flow outside the compact region containing critical points.

Thus, when it comes time to choose microlocal cutoffs, the only important thing is to make sure they are identically constant at the critical points of H_p , and the dynamical condition required for gluing will then be satisfied.

Theorem 4. *Let P have symbol $p(x, \xi) = |\xi|^2 + V_0(x)$, with V_0 as above. Then for any $\epsilon > 0$ we have*

$$\|\rho_s(P - z)u\|_{L^2} \geq Ch^{2m^2/((m-1)(m+2))+\epsilon} \|\rho_{-s}u\|_{L^2}.$$

Proof. Let $\epsilon_0 > 0$. Let C be such that

$$|1 - x| \leq Ch^{2m/((m-1)(m+2))+\epsilon_0/m} \implies |V_0(x) - \tilde{V}(x)| \leq ch^{2m^2/((m-1)(m+2))+\epsilon_0},$$

for some other constant c . Let $\chi \in C_0^\infty(\mathbb{R})$, with $\chi \equiv 1$ on $(-C/2, C/2)$ and $\text{supp } \chi \subset (-C, C)$. Let

$$\chi_h(x) = \chi\left(\frac{x-1}{h^{2m/((m-1)(m+2))+\epsilon_0/m}}\right).$$

Then $\text{supp } \chi_h \subset (1 - Ch^{2m/((m-1)(m+2))+\epsilon_0/m}, 1 + Ch^{2m/((m-1)(m+2))+\epsilon_0/m})$. Now define $V_h = \chi_h(x)\tilde{V}(x) + (1 - \chi_h(x))V_0(x)$. Then

$$|V_h(x) - V_0(x)| \leq ch^{2(m^2/(m-1)(m+2))+\epsilon_0}.$$

Now let P_h have symbol $|\xi|^2 + V_h(x)$. We immediately have

$$\begin{aligned} \|\rho_{-s}(P - P_h)u\|_{L^2} &= \|\rho_{-s}(V_0(x) - V_h(x))u\|_{L^2} \\ &\leq ch^{2m^2/((m-1)(m+2))+\epsilon_0} \|\rho_{-s}u\|_{L^2} \\ &\leq ch^{2m^2/((m-1)(m+2))+\epsilon_0} \|\rho_s u\|_{L^2}. \end{aligned}$$

We will use gluing to show

$$(6.1) \quad \|\rho_s(P_h - z)u\|_{L^2} \geq C \frac{h^{2(m^2/(m-1)(m+2))+\epsilon_0/m}}{\log(1/h)^4} \|\rho_{-s}u\|_{L^2}.$$

Then we may absorb the term which is lower order in h :

$$\begin{aligned}
\|\rho_s(P-z)u\|_{L^2} &= \|\rho_s(P_h-z)u + \rho_s(P-P_h)u\|_{L^2} \\
&\geq \|\rho_s(P_h-z)u\|_{L^2} - \|\rho_s(P-P_h)u\|_{L^2} \\
&\geq c \frac{h^{2(m^2/(m-1)(m+2))+\epsilon_0/m}}{\log(1/h)^4} \|\rho_{-s}u\| \\
&\quad - ch^{2m^2/((m-1)(m+2))+\epsilon_0} \|\rho_{-s}u\| \\
&\geq c \frac{h^{2(m^2/((m-1)(m+2))+\epsilon_0/m)}}{\log(1/h)^2} \|\rho_{-s}u\| \\
&\geq ch^{2(m^2/((m-1)(m+2))+\epsilon_1)} \|\rho_{-s}u\|
\end{aligned}$$

To conclude the proof we prove (6.1). Let $N(h)$ be the number of critical points of V_h . By the above discussion, $N(h) = \mathcal{O}(\log(1/h))$. Let Γ_0 similarly be a microlocal cutoff for the maximum at x_0 , such that

$$\|(P-z)\Gamma_0 u\| \geq C \frac{h}{\log(1/h)} \|\Gamma_0 u\|.$$

Note that the symbol of Γ_0 is independent of h .

Let $\epsilon > 0$ be as in Proposition A.4 and let $\gamma \in C_0^\infty(T^*\mathbb{R})$ have support in $\{|(x, \xi)| < \epsilon\}$ and be such that $\gamma \equiv 1$ on $\{|(x, \xi)| < \epsilon/2\}$. Then define

$$\gamma_j = \gamma((x-x_j)/h^{2m^2/(m-1)(m+2)+\epsilon_0/m}, \xi),$$

so $\gamma_j \in S_{m/(m-1)(m+2)+\epsilon_0/2m}^0$. Let $\Gamma_j = \gamma_j^w$ and by Proposition A.4 we have

$$\|(P-z)\Gamma_j u\| \geq Ch^{2m/(m+2)} \|\Gamma_j u\|.$$

Similarly let $\gamma_\infty = \gamma((x-1)/h^{2m^2/((m-1)(m+2))+\epsilon_0/m}, \xi)$, and let $\Gamma_\infty = \gamma_\infty^w$. Note that the support of each γ_j (including γ_∞) contains only one critical point. Finally, let $\tilde{\Gamma} = 1 - \sum_{j \geq 0} \Gamma_j - \Gamma_\infty$. Since the support of the symbol of $\tilde{\Gamma}$ has no critical points, we have

$$\|(P-z)\tilde{\Gamma}u\| \geq Ch \|\tilde{\Gamma}u\|.$$

We verify that the conditions of Theorem 3 hold. First note that $N(h)$ grows only logarithmically, and as mentioned above the dynamical condition easily holds. The uniform bound condition is satisfied by our microlocal cutoffs because the Γ_j (for $j \geq 1$) are all essentially the same cutoff. Applying gluing, we obtain (6.1). \square

7. FURTHER EXAMPLES

If we change the sequence x_n we obtain a different estimate, with loss depending on how quickly x_n converges. Let $x_n = 1 - a_n$ with $a_n \searrow 0$. The more quickly a_n decreases, the less loss we obtain in our resolvent estimate. We illustrate this with the case

$$a_n = \frac{1}{n^k},$$

with all our inflection points of order m . Let $\alpha_n = 1 + (x_n)^m$, and $\beta_n = 1$ as before. Then

$$d_n = \frac{1}{n^k} - \frac{1}{(n+1)^k} = \frac{(n+1)^k - n^k}{n^k(n+1)^k} \leq \frac{C'}{n^{k+1}},$$

and

$$\alpha_n - \alpha_{n+1} = \frac{1}{n^{km}} - \frac{1}{(n+1)^{km}} = \frac{(n+1)^{km} - n^{km}}{n^{km}(n+1)^{km}} \geq \frac{C}{n^{km+1}},$$

at least for large n . Thus

$$(\alpha_n - \alpha_{n+1}) - (d_n)^m \geq \frac{C}{n^{km+1}} - \frac{C'}{n^{km+m}},$$

so there is again wiggle room to ensure smoothness. Let

$$\tilde{V}(x) = 1 + (1-x)^m,$$

so $\tilde{V}(x_n) = V_0(x_n)$, and as before we have for $x \in [x_n, x_{n+1}]$,

$$|V_0(x) - \tilde{V}(x)| \leq \alpha_n - \alpha_{n+1}.$$

We need

$$\frac{(n+1)^{km} - n^{km}}{n^{km}(n+1)^{km}} \leq Ch^{2m(km+1)/((m+2)(km-k-2))+\epsilon}.$$

For large n , this is just

$$Cn^{-km-1} \leq Ch^{2m(km+1)/((m+2)(km-k-2))+\epsilon},$$

or

$$Cn^{-k-1} \leq Ch^{2m(k+1)/((m+2)(km-k-2))+\epsilon(k+1)/(km+1)}.$$

So we need to consider $N(h) = \mathcal{O}(h^{-2m/((m+2)(km-k-2))-\epsilon/(km+1)})$ critical points, and we will replace V_0 with \tilde{V} when the distance between critical points is smaller than $Ch^{2m(k+1)/((m+2)(km-k-2))+\epsilon(k+1)/(km+1)}$. For large enough k and m , the power of h is less than $2/3$, so we will be able to use gluing. The precise requirement on k and m can be explicitly worked out as

$$\frac{5m+4}{m^2-2m-2} \leq k.$$

We have the following theorem.

Theorem 5. *Let P have symbol $p(x, \xi) = |\xi|^2 + V_0(x)$, with V_0 as above. Then*

$$\|\rho_s(P - z)u\|_{L^2} \geq ch^{2m(km+1)/((m+2)(km-k-2))+\epsilon} \|\rho_{-s}u\|_{L^2}.$$

If

$$\frac{3m+4}{m-2} \leq k,$$

then the power of h is less than 2, and the result may be applied to obtain a local smoothing result on the corresponding manifold. Note that as $k \rightarrow \infty$ we obtain the result of Theorem 4.

For a final example we consider the case of inflection points of alternating order. As before, let $x_n = 1 - 2^{-n}$. Take as our critical values $\alpha_n = 1 + 2^{-5n}$. Near each critical point x_n we have

$$V_0(x) = \alpha_n - \beta_n(x - x_n)^3 - (x - x_n)^5 + \mathcal{O}(|x - x_n|^6),$$

where

$$\beta_n = \begin{cases} 2^{-4n} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Then

$$\begin{aligned}\alpha_{n+1} &= V_0(x_{n+1}) \\ &= \alpha_n + \beta_n(x_{n+1} - x_n)^3 + (x_{n+1} - x_n)^5 + \mathcal{O}(|x_{n+1} - x_n|^6).\end{aligned}$$

We need

$$\alpha_{n+1} - \alpha_n \geq \beta_n d_n^3 + d_n^5.$$

We have

$$\alpha_n - \alpha_{n+1} = 2^{-5n} - 2^{-5n-5} = \frac{31}{32}2^{-5n}.$$

If n is even we have

$$\beta_n d_n^3 = 2^{-2n}(2^{-n-1})^3 = 2^{-5n}2^{-3} = \frac{1}{8}2^{-5n},$$

and in either case,

$$d_n^5 = 2^{-5n-5} = \frac{1}{32}2^{-5n}.$$

So in the worst case scenario we have

$$\frac{31}{32}2^{-5n} \geq \frac{5}{32}2^{-5n},$$

which gives us enough wiggle room to ensure V_0 is smooth.

Now let $\tilde{V}(x) = 1 + (1 - x)^5$. Then as before we have, for $x \in [x_n, x_{n+1}]$,

$$|V_0(x) - \tilde{V}(x)| \leq \alpha_n - \alpha_{n+1} \leq c2^{-5n},$$

so to obtain a difference of less than $ch^{50/28+\epsilon}$, we must have

$$2^{-5n} \leq ch^{50/28+\epsilon},$$

or

$$2^{-n} \leq ch^{10/28+\epsilon/5}.$$

Thus we cutoff V_0 on scales $h^{10/28+\epsilon/5}$, and consider $\mathcal{O}(\log(1/h))$ critical points.

Repeating the argument of Theorem 4, we obtain

Theorem 6. *Let P have symbol $p(x, \xi) = |\xi|^2 + V_0(x)$, with V_0 as above. Then*

$$\|\rho_s(P - z)u\|_{L^2} \geq ch^{50/28+\epsilon} \|\rho_{-s}u\|_{L^2}.$$

Note that the gluing lemma requires $N(h)$ to grow more slowly than some polynomial, and so does not apply in the case where x_n behaves like $\frac{1}{\log n}$. Heuristically we would expect a loss of $2^{1/h}$, or in other words no smoothing. Of course, it is unknown whether the results obtained here are sharp.

APPENDIX A. A CATALOG OF RESOLVENT ESTIMATES

Our manifold will have no stable trapping, so all trapping is unstable, consisting of disjoint critical sets, and even if two critical sets exist at the same potential energy level, they must be separated by an unstable maximum critical value at a *higher* potential energy level (otherwise there would be a minimum in between, and hence at least weakly stable trapping), so they do not see each other. That is to say, the weakly stable/unstable manifolds of the separating maximum form a separatrix in the reduced phase space. This allows us to glue together microlocal estimates near each critical set, and the resolvent estimate is then simply the worst of these estimates. Hence it suffices to classify microlocal resolvent estimates in

a neighbourhood of any of these unstable critical sets. This is accomplished in Subsections A.1-A.3. In this sense, this section contains a catalogue of microlocal resolvent estimates.

It is important to note at this point that for unstable trapping of finite degeneracy, the relevant resolvent estimates are all $o(h^{-2})$, that is to say, the sub-potential h^2V_1 is always of lower order.

These results may be used in conjunction with the above methods to obtain resolvent estimates on manifolds with a trapped set with infinitely many connected components by taking the worst resolvent estimate and taking into account any loss coming from gluing an infinite number of resolvent estimates.

A.1. Unstable nondegenerate trapping. Unstable nondegenerate trapping occurs when the potential V_0 has a nondegenerate maximum. As mentioned previously, let us for the time being consider the operator $\tilde{Q} = -h^2\partial_x^2 + V_0(x) - z$, where $V_0(x) = A^{-2}(x)$. To say that $x = 0$ is a nondegenerate maximum means that $x = 0$ is a critical point of $V_0(x)$ satisfying $V_0'(0) = 0$, $V_0''(0) < 0$, and then the Hamiltonian flow of $\tilde{q} = \xi^2 + V_0(x)$ near $(0, 0)$ is

$$\begin{cases} \dot{x} = 2\xi, \\ \dot{\xi} = -V_0'(x) \sim x, \end{cases}$$

so that the stable/unstable manifolds for the flow are transversal at the critical point $(0, 0)$.

The following result as stated can be read off from [Chr07, Chr10, Chr11], and has also been studied in slightly different contexts in [CdVP94a, CdVP94b] and [BZ04], amongst many others. We only pause briefly to remark that, since the lower bound on the operator \tilde{Q} is of the order $h/\log(1/h) \gg h^2$, the same result applies equally well to $\tilde{Q} + h^2V_1$.

Proposition A.1. *Suppose $x = 0$ is a nondegenerate local maximum of the potential V_0 , $V_0(0) = 1$. For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x, \xi)| \leq \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$(A.1) \quad \|\tilde{Q}\varphi^w u\| \geq C_\epsilon \frac{h}{\log(1/h)} \|\varphi^w u\|, \quad z \in [1 - \epsilon, 1 + \epsilon].$$

A.2. Unstable finitely degenerate trapping. In this subsection, we consider an isolated critical point leading to unstable but finitely degenerate trapping. That is, we now assume that $x = 0$ is a degenerate maximum for the function $V_0(x) = A^{-2}(x)$ of order $m \geq 2$. If we again assume $V_0(0) = 1$, then this means that near $x = 0$, $V_0(x) \sim 1 - x^{2m}$. Critical points of this form were studied in [CW13]. We only remark briefly that again, since the lower bound on the operator \tilde{Q} is of the order $h^{2m/(m+1)} \gg h^2$, the estimate applies equally well to $\tilde{Q} + h^2V_1$.

Proposition A.2. *Let $\tilde{Q} = -h^2\partial_x^2 + V_0(x) - z$. For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x, \xi)| \leq \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$(A.2) \quad \|\tilde{Q}\varphi^w u\| \geq C_\epsilon h^{2m/(m+1)} \|\varphi^w u\|, \quad z \in [1 - \epsilon, 1 + \epsilon].$$

Remark A.3. In [CW13], it is also shown that this estimate is sharp in the sense that the exponent $2m/(m+1)$ cannot be improved.

A.3. Finitely degenerate inflection transmission trapping. We next study the case when the potential has an inflection point of finitely degenerate type. That is, let us assume the point $x = 1$ is a finitely degenerate inflection point, so that locally near $x = 1$, the potential $V_0(x) = A^{-2}(x)$ takes the form

$$V_0(x) \sim C_1^{-1} - c_2(x-1)^{2m_2+1}, \quad m_2 \geq 1$$

where $C_1 > 1$ and $c_2 > 0$. Of course the constants are arbitrary (chosen to agree with those in [CM14]), and c_2 could be negative without changing much of the analysis. This Proposition and the proof are in [CM14]. One last time, let us observe that since the lower bound on the operator \tilde{Q} is of the order $h^{(4m_2+2)/(2m_2+3)} \gg h^2$, the estimate applies equally well to the operator $\tilde{Q} + h^2V_1$.

Proposition A.4. *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x-1, \xi)| \leq \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$(A.3) \quad \|\tilde{Q}\varphi^w u\| \geq C_\epsilon h^{(4m_2+2)/(2m_2+3)} \|\varphi^w u\|, \quad z \in [C_1^{-1} - \epsilon, C_1^{-1} + \epsilon].$$

Remark A.5. We remark that in this case, [CM14] shows once again that this estimate is sharp in the sense that the exponent $(4m_2 + 2)/(2m_2 + 3)$ cannot be improved.

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